Departamento de Física Teórica de la
Universidad Autónoma de Madrid.

THE BRADLOW PARAMETER EXPANSION AND ITS APPLICATIONS IN FIELD THEORY.

Memoria de Tesis doctoral presentada ante la facultad de ciencias, sección de ciencias físicas, de la Universidad Autónoma de Madrid, por

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para la obtención del titulo de Doctor en Ciencias Físicas.

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Resumen†

El el capitulo 2 se realiza una breve introducción a distintos aspectos de la física no perturbativa. Empezando con los limites de las descripciones perturbativas de las teórias de campos, se pondrán ejemplos de contribuciones no perturbativas en modelos de juguete basados en la mecánica cuántica, pero útiles para captar la esencia de los problemas a los que nos enfrentamos. Mas tarde pasaremos a estudiar estos efectos no perturbativos en QCD, donde básicamente tenemos dos fenómenos no perturbativos que caracterizan la teoría: La ruptura espontánea de la simetría quiral y el confinamiento. La relación de estos problemas con las soluciones clásicas sera explorada, con la intención de que se entienda la necesidad de una buena comprensión de las soluciones clásicas: Instantones, monopolos, vórtices,etc. . . para entender los efectos no perturbativos en teoría de campos. El capitulo tambien incluye una breve introducción a los campos gauge en el toro, así como unos comentarios sobre la importancia que estas soluciones clásicas de tipo solitón juegan en el concepto de dualidad, sobre todo en teorías supersimetricas. La idea esencial a sacar de este capitulo es la siguiente: una buena comprensión de los efectos no perturbativos en teoría de campos es deseable conocer o bien en forma exacta, o bien aproximadamente la forma de las soluciones clásicas de la teoría. Desgraciadamente conocer la forma de estas soluciones normalmente requiere resolver ecuaciones diferenciales no lineales en derivadas parciales, algo que es un arte en si mismo. Cualquier tipo de solución aproximada o método para solucionar algunos de estos tipos de ecuaciones es bienvenido. Por ultimo tendremos un breve encuentro con las idea principal de la tesis: la expansión en el parámetro de Bradlow. La clave de la idea consiste en que si definimos la teoría en una variedad compacta, podemos aprovechar que algunas soluciones clásicas se conocen para ciertos valores del volumen de la variedad (volumen critico). Una solución general, para volumen arbitrario, se construye perturbando alrededor de esa solución conocida, y se obtiene en forma de serie de potencias en un parámetro que interpola entre el volumen para el cual conocemos las soluciones y el caso de volumen infinito. Este es el parámetro de Bradlow.

En el siguiente capitulo estudiaremos en detalle el modelo Higgs abeliano y sus soluciones clásicas: los vórtices de Nielsen-Olesen. Allí repasaremos las propiedades que conocemos de estos objetos: su comportamiento asintótico, una expansion para obtener las soluciones con

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simetría cilíndrica, la estructura del espacio de soluciones, etc... Encontraremos las ecuaciones d Bogomolny, un conjunto de ecuaciones diferenciales no lineales en derivadas parciales de primer orden, cuyas soluciones coinciden con las soluciones de las ecuaciones de movimiento para cierto valor de la constante de acoplo del Higgs con el mismo. Como veremos ese valor de la constante de acoplo es precisamente el mas interesante, y las ecuaciones de Bogomolny jugaran el mismo papel que las ecuaciones de autodualidad en el caso de los instantones. En la ultima parte del capitulo presentaremos el modelo Higgs abeliano extendido: una generalización del modelo Higgs abeliano, donde tenemos varios campos escalares. Este tipo de modelos extendidos son muy interesantes, porque aparecen de forma natural como la parte bosonica de teorías gauge supersimetricas [1, 2, 3, 4].

En el capitulo 4 estudiaremos el modelo Higgs abeliano y sus extensiones en el toro bidimensional. La clave esta en estudiar los campos gauge abelianos en el toro. Aunque para ese entonces ya hemos hablado de los campos gauge en el toro en la sección 2.6, el hecho de estar tratando con un campo abeliano hace que el tema merezca un lugar propio. La forma de las ecuaciones de Bogomolny, así como la caracterización del espacio de soluciones, tanto para el modelo Higgs abeliano, como para los modelos extendidos ocupara una parte esencial del capitulo.

El capitulo 5 esta dedicado a hallar soluciones a las ecuaciones de Bogomolny. Aquí es donde aparece el tema fundamental de la tesis: aplicaremos la expansion en el parámetro de Bradlow para obtener la forma exacta de las soluciones. El primer orden en la expansión, que resolvemos de forma analítica exacta para cualquier valor del flujo y cualesquiera posiciones de los vórtices, se puede utilizar como una soluciona aproximada, valida cuando el área es cercana al area critica. Con ayuda de un ordenador seremos capaces de calcular 51 ordenes en esta expansion, siendo asi capaces de chequear la convergencia de la serie, extrapolar los valores en el toro para obtener vórtices en el plano, y por ultimo aplicar el mismo esquema para hallar soluciones al modelo Higgs abeliano extendido.

El capitulo 6 esta dedicado a estudiar la métrica en el espacio de soluciones de las ecuaciones de Bogomolny. Esta metrica, inducida por la energía cinética, nos da una aproximación al problema de la colisión de vórtices, como el movimiento de partículas puntuales en una variedad curvada, dotada de la métrica antes mencionada. La comprensión de la forma de este fenómeno puede ser crucial para el mundo de la cosmología, por su relevancia en relación con las cuerdas cósmicas. Como veremos en este capitulo, no existe una forma explicita para la métrica, solamente se conoce su forma asintótica para vórtices muy separados.

En el capitulo 7 estudiaremos la métrica para el caso de vórtices viviendo en el toro. Obtendremos una formula para calcular la métrica, y analizaremos las simetrías que posee. El mismo análisis se realizara para el caso de la métrica en el espacio de soluciones de los modelos Higgs abelianos extendidos.

En el capitulo 8 aplicaremos el método de la expansión en el parámetro de Bradlow al análisis de la métrica. Demostraremos que la métrica se puede escribir como una serie de potencias en el parámetro de Bradlow, y calcularemos explicitamente los dos primeros ordenes para un numero arbitrario de vórtices. Para cada orden finito, demostraremos que la métrica se puede escribir en función de un numero finito (y pequeño) de parámetros. El caso especial de la dinámica de dos vórtices sera analizado en detalle. Determinaremos con precisión maquina la métrica hasta quinto orden y pintaremos la forma de los mapas de Poincare, preguntándonos sobre la integrabilidad del problema. Calculando 40 ordenes extrapolaremos nuestros datos para obtener la métrica de los vórtices viviendo en $\mathbb{R}^2$. Compararemos esta métrica con lo que sabemos de la métrica de los vórtices en el plano. Como veremos el acuerdo es mas que satisfactorio. Esta métrica se usara para determinar la dinámica de las colisiones de dos vórtices, obteniendo el ángulo de dispersión como función del parámetro de impacto.

Por ultimo en el capitulo 9 se mostrara un trabajo en curso. La idea es aplicar la expansión en el parámetro de Bradlow para analizar el comportamiento de los fermiones en presencia de los vórtices. Demostraremos que los autovalores y autofunciones se pueden obtener como una serie de potencias en este parámetro, y calcularemos explícitamente la forma del orden mas bajo. Los ordenes mas altos necesitan de un calculo por ordenador, que en el momento de escribir este trabajo esta siendo desarrollado.

Los apéndices de este trabajo son una parte importante del mismo. El apéndice A es un pequeño resumen de las notaciones y convenios que tomamos a lo largo del texto. El apéndice B es un "curso breve" autocontenido en funciones Theta: sus propiedades mas importantes serán estudiadas, ya que son unas funciones que aparecerán en distintas partes del texto. El apéndice C esta dedicado al estudio de un espacio de Hilbert especial, formado por los campos que cumplen ciertas propiedades de quasi periodicidad, que son el equivalente abeliano de las condiciones de frontera "twisted". Encontraremos una base en este espacio, que usaremos en distintas partes del texto, así como sus propiedades. El apéndice D esta dedicado a desarrollar una técnica que permita obtener los autovalores de una matriz dada como una serie de potencias en un parámetro. Estas tecnicas se usaran para estudiar el comportamiento de los fermiones en presencia de vórtices, y para otros trabajos que actualmente están bajo estudio, como es el calculo de las correcciones cuánticas a las soluciones tipo vórtice. El ultimo apéndice E, es el "manual" de usuario del código en `FORTRAN 90` que hemos usado para obtener todas las soluciones numéricas a lo largo del texto. En particular hay ejemplos de como calcular las soluciones al modelo Higgs abeliano, a los modelos extendidos, o como calcular la métrica en el espacio de soluciones. Todo el código es software libre, distribuido bala licencia `GPL`[1]. Esto significa, entre otras cosas, que puede usar el codigo todas las veces que quiera para obtener sus propios resultados. El codigo hace uso de la librería numérica `afnl`[2], que puede considerarse un subproducto de esta tesis.

[1] `http://www.gnu.org/copyleft/gpl.html`

[2] El codigo fuente de esta libreria y su documentación se puede encontrar en la pagina de su proyecto en spurceforge: `http://sourceforge.net/projects/afnl`

Two

Introduction[†]

> There has been sensational progress in *calculating* quantum electrodynamics, but very little progress in *understanding* it; and strong interactions are neither calculable nor understood.
>
> K. G. Wilson and J. Kogut

Quantum field theory is nothing but the aim to put the old quantum mechanics and the special theory of relativity under a consistent framework, thus it is natural that it is at the core of modern physics: It is one of the two pillars under which our understanding of nature relies, together with Einstein's general theory of relativity. The standard model (a particular quantum field theory), that is able to describe the weak, strong and electromagnetic interactions, is usually seen as the most precise theory ever tested: almost all the experiments done up to the present date agree with the theoretical predictions. The overwhelming accuracy of the theory in some experiments (1 part in 10^{12}) makes difficult to have doubts about the correctness of the model as an accurate description of the particle physics up to energies of 1 TeV. The small discrepancies between theory and experiment are always attributed to the particular model, not to problems of the quantum field theory.

The only point in which quantum field theory has not succeeded is in describing the gravitational force, but this is another story.

Under this impressive curriculum, that is usually the "presentation card" of quantum field theory, there are lot of common day phenomena that still need an explanation and a qualitative understanding. Almost all of the theoretical predictions that agree so well with the experiments, come from our perturbative understanding of quantum field theories. These are the predictions so well tested at CERN and other laboratories. But non perturbative quantum field theory is, by far, a very different story. Only now, thanks to the enormous advance in recent years of the Lattice formulation of gauge theories, and to the impressive computational power that modern computers have, we are able to compute some non perturbative effects

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for realistic models. Even so, these non perturbative effects are central for the understanding of the world in which we live: They should explain why protons and neutrons form, and do not break into quarks, why they remain together in the nuclei and almost all the matter that we see is as we see it. In spite of the high precision tests of the standard model, and the unquestionable fact that this model should explain the properties of the matter that is around us, our poor understanding of the non perturbative effects of strongly coupled quantum field theories make impossible to get a qualitative understanding of these problems.

This introductory chapter aims to be a small trip through some non perturbative effects of quantum field theory. This is a vast subject, and tons of books have been devoted to the subject. Here we will only sketch some problems selected by a personal point of view.

2.1 Perturbative quantum field theory

As an example of a quantum field theory that can be described perturbatively, we will use quantum electrodynamics (QED). This is the theory that describes the interaction of electrons and photons. The electrons are described by a spin 1/2 field

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \tag{2.1}$$

and the photons by a spin 1 gauge field A_μ. The theory has a local $U(1)$ symmetry, and in quantum field theory this defines uniquely a theory: The matter content and the symmetries. The Lagrangian of QED is

$$\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi} \left(\imath \gamma^\mu D_\mu - m_e \right) \psi \tag{2.2}$$

where γ_μ are Dirac matrices, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength, $D_\mu = \partial_\mu - \imath e A_\mu$ is the covariant derivative, that gives the coupling between electrons and photons, and m_e and e are the electron mass and charge. The interaction term between electrons and photons, gives us the interaction Hamiltonian

$$\mathcal{H}_I = -e \overline{\psi} \gamma^\mu A_\mu \psi \tag{2.3}$$

that leads to the Dyson expansion of the S-matrix

$$S = \mathrm{T} \exp \left\{ -\imath \int d^4 x \, \mathcal{H}_I(x) \right\} \tag{2.4}$$

Using this expansion we can deduce the Feynman rules that allows us to compute perturbatively any process that we want as a power series in the fine structure constant $\alpha_e = e^2/(4\pi) \approx 1/137$

$$\left\langle \phi_{\text{in}} | \phi'_{\text{in}} \right\rangle = \left\langle \phi_{\text{in}} \mid S \mid \phi'_{\text{out}} \right\rangle = \sum_{n=0}^{\infty} a_n \alpha_e^n \tag{2.5}$$

Loosely speaking this is the way in which all the perturbative processes are computed. The procedure to define and compute the terms in the series is by no means a trivial task: Loop

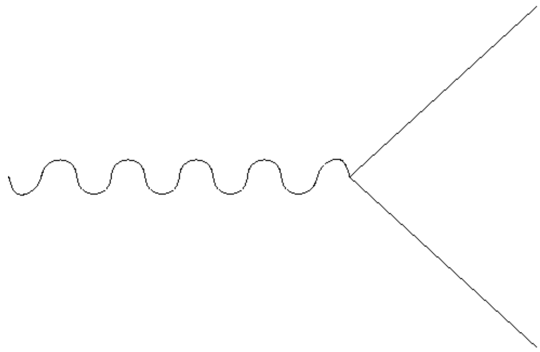


Figure 2.1: The fundamental vertex of QED.

contributions give naively an infinite contribution, that has to be *renormalised*, by a procedure that is now well understood, to obtain a finite and meaningful contribution to the power series. Although the practical problems of achieving this goal can be enormous, there is no fundamental problem in doing it. The small value of the fine structure constant, help us, because with only a handful of terms of the series we should obtain a good approximation to the full computation.

This perturbative series presents a coherent picture of the interactions between the particles that we "see": electrons and photons. Through the fundamental interaction given by the Hamiltonian of equation (2.3), that can be represented by the vertex of the figure (2.1) lot of common-day phenomena can be easily interpreted (for a really magical exposition of some of them, read [5]).

2.1.1 The convergence of the perturbative series

It was first noted by Dyson that the perturbative series of quantum electrodynamics are, in general not convergent. The simple, but deep observation that help us to conclude this is that if we reverse the sign of the fine structure constant α_e, the vacuum of QED is unstable under the emission of pairs of electrons and positrons, thus spoiling the convergence of the power series. But it is a well known fact of complex analysis that a power series cannot be convergent for any positive value of α_e without being convergent for any negative value.

After having commented the great success of the perturbative description of the standard model in explaining the experiments of the particle accelerators like LEP, it is difficult to believe that these series are incorrect, and do not have any predictive power in spite of the convergence properties of the series.

Being a non convergent series, does not mean, in principle that the expansion is useless. The reason for that are the existence of the asymptotic expansions. To have an explicit example, imagine that we face the series

$$\sum_n (-1)^n n!\, x^{n+1} \tag{2.6}$$

since

$$\left| \frac{(-1)^{n+1}(n+1)! x^{n+2}}{(-1)^n n! x^{n+1}} \right| = (n+1)x \underset{n\to\infty}{\longrightarrow} \infty \tag{2.7}$$

the series is clearly divergent for all values of x (i.e. with zero radius of convergence). This series can be seen as an expansion for the function

$$f(x) = \int_0^\infty \frac{e^{-t/x}}{1+t}\, dt \tag{2.8}$$

after making an expansion of the fraction in powers of t. Still the divergent series (2.6) is useful for computing values of $f(x)$ for values of x close to zero. To see why we define the partial sum of the series

$$S_N(x) = \sum_{n=0}^{N} (-1)^n n!\; x^{n+1} \tag{2.9}$$

the approximation to the value of the function that this partial sum gives is

$$R_N(x) = |f(x) - S_N(x)| = \int_0^\infty e^{-t/x} \frac{t^{N+2}}{1+t}\, dt \tag{2.10}$$

The fact that the series has a zero radius of convergence is reflected by the property

$$\lim_{N\to\infty} R_N(x) = \infty \qquad \forall x \setminus \{0\} \tag{2.11a}$$

still, it can be easily proven (by integration by parts), that we can obtain good approximations to the values of $f(x)$ using the series for values of x close to zero, because

$$R_N(x) < N! x^{N+1} \underset{x\to 0}{\longrightarrow} 0 \tag{2.11b}$$

in this sense our series can be very useful to compute values of $f(x)$ close to $x = 0$. This is the basic concept of an asymptotic series[1]: it is a non convergent series, but the partial sums of the terms of the series are a good approximation to the value of the function close to certain point. To say that our series is, in fact, an asymptotic series of $f(x)$, we write

$$f(x) \sim \sum_n (-1)^n n!\, x^{n+1} \tag{2.12}$$

We can see a practical example of the use of our asymptotic series: to compute the value

$$f(0.2) = 1.70\ 42\ 21\ 76\ 28 \times 10^{-1} \tag{2.13a}$$

we see that with two terms of the series we obtain the value

$$S_2(0.2) = 1.60\ 00\ 00\ 00\ 00 \times 10^{-1} \tag{2.13b}$$

the best approximation to the value of $f(0.2)$ is attained summing the first 5 terms of the series

$$S_5(0.2) = 1.66\ 40\ 00\ 00\ 00 \times 10^{-1} \tag{2.13c}$$

adding more terms does not improve the computation of the value of the function, but only makes things worse. Here is where the non convergent character of the series manifest (see figure (2.2)).

[1]The formal definition of an asymptotic series, due to Poincare, is more restrictive than the property that we have mentioned on the text. For a detailed description of the definition and properties of the asymptotic series the reader should consult [6, Chapter VIII] or [7]. Our example also satisfies the formal definition of asymptotic series given in the cited references.

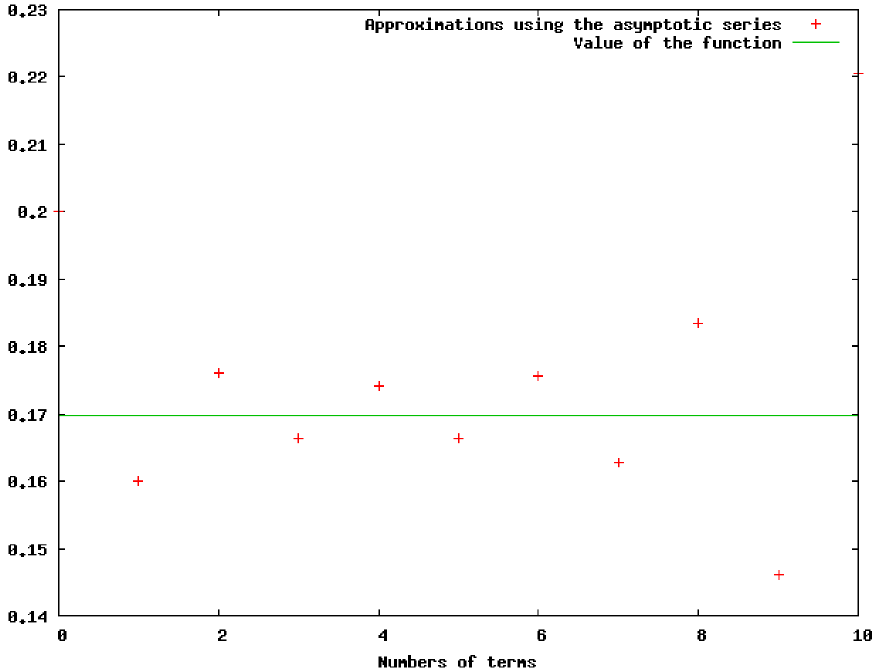


Figure 2.2: A comparison between the partial sums of the asymptotic series (2.6) and the value of the function $f(x)$ defined in (2.8) in the point $x = 0.2$.

In fact our function $f(x)$ of equation (2.8) can be related to the first exponential integral, defined by

$$E_1(x) = \int_x^{\infty} \frac{e^{-t}}{t}\, dt \tag{2.14}$$

the relation is

$$E_1(x) = e^{-x} f\left(\frac{1}{x}\right) \tag{2.15}$$

and then our original asymptotic expansion of equation (2.6) gives the following asymptotic expansion for the first exponential integral

$$E_1(x) \underset{x\to\infty}{\longrightarrow} \frac{e^{-x}}{x} \sum_{n=0}^{\infty} \frac{(-1)^n n!}{x^n} \tag{2.16}$$

This asymptotic expansion for the first exponential integral can be found in lot of textbooks, for example [8].

Asymptotic series are very common: everybody knows the Stirling approximation of the factorial, and almost any "special" function has an asymptotic expansion useful to compute it at some point (often at infinity). In fact, most computer software and calculators, use asymptotic series at some point to compute the value of some special functions.

The problem with the asymptotic series, is that although it's trivial to prove (see [6] for details) that one function can not have two different asymptotic expansions, it is equally trivial to show that two different functions can have the same asymptotic expansion. To show this it is enough to notice that we can add to our function $f(x)$ any analytic function $g(x)$ such that[2]

$$\lim_{x\to 0} \frac{g(x)}{x^n} = 0 \qquad \forall n \tag{2.17}$$

and the asymptotic expansion of $f(x)$ is as well an asymptotic expansion of $f(x)+g(x)$. Another way to say the same thing is that an asymptotic expansion gives us partial information of a function $f(x)$, that has to be completed with more information about the properties of the function.

The problem can be stated in the following way: since the perturbative series in quantum field theory are not convergent series, they should be asymptotic series in order to explain why the computation of a couple of orders gives a good approximation to the experimental quantities, and we want to know under which conditions these series, or these series with other information that we can obtain from the theory, can determine completely the observable quantities: we want to reconstruct the function $f(x)$ from its asymptotic expansion. There is one situation when an asymptotic series defines a function in a unambiguous way: when we can *Borel sum* it. The procedure is as follows: if we have the divergent series of a unknown function $f(x)$

$$f(x) \sim \sum_n a_n \, x^n \tag{2.18}$$

we define a related series, where each coefficient is obtained dividing the terms of the previous series by a $(n-1)!$ term

$$\sum_n \frac{a_n}{(n-1)!} \, x^{n-1} \tag{2.19}$$

Clearly this series has more chances of being convergent. If this series has a finite radius of convergence, then by analytic continuation, it unambiguously defines a function $\tilde{f}(x)$

$$\tilde{f}(x) = \sum_n \frac{a_n}{(n-1)!} \, x^{n-1} \tag{2.20}$$

Now we define the original function to be the Laplace transform of $\tilde{f}(x)$ evaluated at the inverse argument, i.e.

$$f(x) = \int_0^\infty e^{-t/x} \tilde{f}(t) \, dt \tag{2.21}$$

In simple words, we have divided and multiplied by a $(n-1)!$ term. The reader can easily check that if we follow this procedure on any convergent series we recover the original function, and if we do that to the divergent series of our original example we automatically obtain the original expression for the function $f(x)$ (equation 2.8). Not any divergent series can be summed with this trick. If the function $\tilde{f}(x)$ has poles in the positive real axis, or grows very fast, its Laplace transform is ill defined, and then our summation method fails. The question on when the asymptotic perturbative series of a theory is "Borel summable" is related to the non perturbative properties of the theory. The necessary and sufficient conditions for this situation are not known in general, but making a long story short, the theory must be free of instanton solutions if we want to have any chance of Borel sum the series. Later we will see in detail what these instanton solutions are, but here we should have learned an important lesson: the

[2] The reason why we need the $1/x^n$ relies in the formal definition of asymptotic expansion. For details see [6, Chapter VIII] or [7]

perturbative series is not convergent, but at best an asymptotic series that can be summed, and the information to perform this summation is in the "dark side" of the theory: its non perturbative properties.

Notes and references

It is impossible to end this section without some references. We have sketched a very difficult problem, and some solutions that have been applied successfully only for some academical models, mainly the $\lambda\phi^4$ theory in two and three dimensions: this fact should give us a hint of the difficulties that we are facing. From my personal point of view, the most comprehensive place to look at these problems is Zinn-Justin's book [9]. The subject of divergent series in mathematics is old, a couple of pages summarising the different methods to sum divergent series, and the most important theorems can be found at [6, Chapter VIII], but as far as I know, *the* reference book in this subject is Hardy's book [7].

Whereas the convergence problem is not the only problem of perturbative series, and the reader that consult [9] will note it, the *triviality* problem is from a personal point of view one of the most important conceptual problems. In the process of renormalizing a quantum field theory, it can happen that the only consistent solution is to switch off the interaction and obtain a non interacting theory in the continuum. This problem again cannot be seen when the relation of the bare and renormalised couplings is given by a perturbative series: we will always obtain an interacting theory in the continuum limit to all orders in perturbation theory. But something can be true to all orders in perturbation theory and be false for the full series, simple examples of this is trying to find a solution to the equation $\sin x = 2$, that, although is a useless work, has a solution to all orders in perturbation theory. The problem of finding an interacting theory in the continuum is hard, but numerical simulations tells us that it seems impossible for the $\lambda\phi^4$ theory in four dimensions, and for sure that this is the case for the same theory with N scalar fields coupled with a $O(N)$ symmetry in the large N limit. This is a serious non academical problem because this field theory is just the scalar sector of the standard model. Of course the addition of new matter content can change this behaviour, but it can also happen that a quantum field theory of fundamental interacting scalars has no sense, or that all the non asymptotically free field theories suffer this problem. A solution of these problems can be very important, because it points where we have to look for candidates of physics beyond the standard model, for example if field theories of self interacting scalar fields do not have a well defined continuum limit technicolor models, in which the Higgs field is not a fundamental field, can be favoured. Once more, a good place to study this topic is [9]. For a clear explanation on how the numerical simulations are done, and some results, a good starting point is [10].

2.2 An example of non perturbative contribution

Here we will briefly show an example of a non perturbative contribution in quantum mechanics. Quantum mechanics can be seen as a quantum field theory in one dimension, being one of the simplest examples that we can imagine. The example that we are going to use is the anharmonic oscillator, with Hamiltonian

$$H = \frac{p^2}{2} - \frac{x^2}{2} + \frac{\lambda^2}{4}x^4 \tag{2.22}$$

the form of the potential can be seen in the figure (2.3). A usual academic exercise consists in

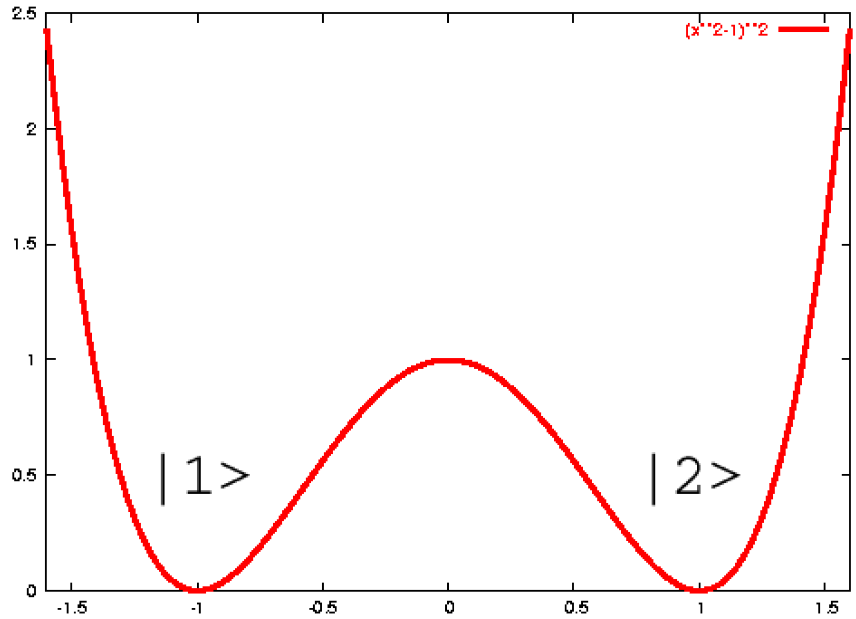


Figure 2.3: Potential of the anharmonic oscillator

finding the corrections to the energy levels as perturbative series in the parameter λ

$$E^{(i)} = \sum_n a_n^{(i)} \lambda^n \tag{2.23}$$

This power series faces the same problems as the general perturbative series of quantum field theory: If we change the sign of λ, the potential is not bounded from below, and then the vacuum is unstable, spoiling the convergence of the series for any positive value of λ, no matter how small. Here we will assume that we know all the coefficients of the perturbative expansion $a_n^{(i)}$, and try to obtain conclusions about the energy levels. First we have two possibilities for expanding the potential, one for each minima of the potential. We can choose any of them, and then our first conclusion arises. All the energy levels are degenerate. The second conclusion is that the parity symmetry is spontaneously broken, because the minimum energy states are not invariant under the transformation $x \to -x$. It is easy to prove that these properties are true to any order in perturbation theory.

No we can use the WKB method to compute the probability of tunnelling between the two minima. It can be shown that if we call $\mid 1\rangle$ the state of a particle oscillating in the minima located at $x = -1/\lambda$, and $\mid 2\rangle$ the state of a particle oscillating in the minima located at $x = 1/\lambda$, the probability amplitude for the transition from the state $\mid 2\rangle$ to the state $\mid 1\rangle$ is given by

$$\langle 1|2\rangle \propto \exp\left\{-S_{\text{euc}}\right\} \tag{2.24}$$

where S_{euc} is the Euclidean action computed along a classical trajectory $\overline{x}(t)$ that goes from one maxima of $-V(x)$ to the other, as can be seen in the figure (2.4).

$$S_{\text{euc}} = \int dt\, L_{\text{euc}} = \int dt\, \left(\frac{1}{2}m\dot{\overline{x}}^2(t) + V\left(\overline{x}(t)\right)\right) \tag{2.25}$$

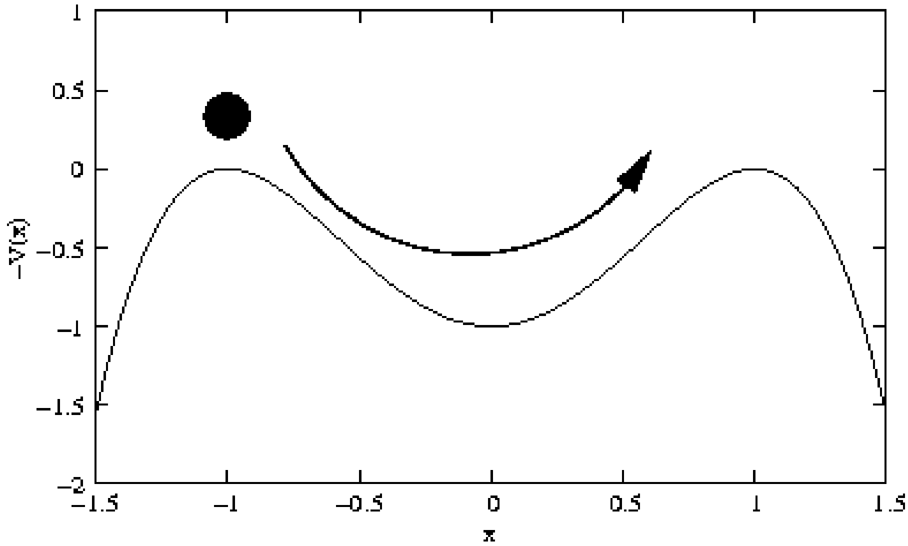


Figure 2.4: Classical trajectory of a particle from one minima to another of $-V(x)$

These solutions of the Euclidean equations of motion, that help us to compute the tunnelling amplitudes are usually called instantons. Since we are interested only in the dependence of this amplitude with the coupling constant λ, it is not necessary to compute explicitly this action. It is enough to note that this dependence is given by

$$\langle 1|2\rangle \propto \exp\left\{-\frac{A}{\lambda^2}\right\} \tag{2.26}$$

This dependence is what we call a non perturbative effect, because this contribution has an essential singularity at $\lambda = 0$, then this contribution cannot be approximated by any power expansion in λ (it falls faster than any power of λ). These kind of contributions are genuinely non perturbative, and this is the reason why we can never see a barrier penetration in perturbation theory. Now it should be clear why the presence of instantons spoil all the possibilities of reconstructing the full functions from it's asymptotic series, as we mentioned in the previous chapter, this contribution are never seen in a power expansion. Another important point to note is that the contribution decreases to zero very fast when the coupling is small.

This barrier penetration, that allows a particle that is oscillating in the state $|\,1\rangle$, to tunnel and oscillate in the state $|\,2\rangle$, automatically tells us that these states are not stationary, the real stationary states of the system are

$$|\pm\rangle = \frac{1}{\sqrt{2}}\left(|\,1\rangle \pm |\,2\rangle\right) \tag{2.27}$$

and are states in which the particle is some time oscillating in one minima, then "jumps" to the other and oscillates there some time, then jumps back, etc... In fact the real ground state is $|+\rangle$, that has an energy

$$E_+ = E_0 - \exp\left\{-\frac{A}{2\lambda^2}\right\} \tag{2.28}$$

now there exist an energy difference between the states that were degenerate in perturbation theory, given by

$$E_+ - E_- = -2\exp\left\{-\frac{A}{2\lambda^2}\right\} \tag{2.29}$$

It is easy to see that the ground state is invariant under a transformation $x \to -x$, so parity is not spontaneously broken.

The picture of a man that makes a non perturbative analysis of this anharmonic oscillator is drastically different: There are no degenerate states, and the system has parity invariance. We have two important lessons in this example: first, the tunnelling between “ground states” is a non perturbative effect, that can never be computed in perturbation theory. They should be computed using instanton contributions. Second, non perturbative effects, when present, drastically change the image that we have of the world, symmetries can be broken or unbroken by its effects, and they can break or restore the degeneracy of the energy levels. We have learned these lessons from a quantum mechanical example, but these conclusions are still true in quantum field theory.

Notes and references

The most clear discussion of these concepts that I know is [11, Chapter 7]. There the tunnelling between different vacua for the case of the anharmonic oscillator is studied in detail with the same tools that are needed to tackle the problem in quantum field theory, later this formalism is used to explain a solution to the $U(1)$ problem. In [9] the anharmonic quantum oscillator is also used as a toy model for studying instantons in gauge theories.

2.3 The strong interactions

The strong force is described by a $SU(3)$ gauge theory. The gauge fields living in the adjoint representation

$$A_\mu = A_\mu^a \lambda_a \tag{2.30}$$

are the gluons and carry the interaction between charged (i.e. coloured) particles. Here λ_a are the generators of the $SU(3)$ Lie algebra, with the normalisation convention

$$\mathrm{Tr}(\lambda_a\lambda_b) = \frac{\delta_{ab}}{2} \tag{2.31a}$$

$$[\lambda_a, \lambda_b] = \imath f_{abc}\lambda_c \tag{2.31b}$$

the field strength is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \imath[A_\mu, A_\nu] \tag{2.32}$$

The fundamental particles that feel the strong force are called *quarks*. These are fields living in the fundamental representation of $SU(3)$. We are going to study a QCD model with the two lightest quarks (the u and d quarks), since more than 99% of the matter that is around us depends only on the properties of these quarks, the simplification is very good to study the properties of matter. As we have mentioned before matter content and symmetries almost define unambiguously the quantum field theory, thus the Lagrangian density of QCD is given by

$$\mathcal{L}_{QCD} = -\frac{1}{2}\mathrm{Tr}\left\{F_{\mu\nu}F^{\mu\nu}\right\} + \sum_{i=u,d} \overline{q}_i(\imath\gamma^\mu D_\mu - m_i)q_i \tag{2.33}$$

Here q_u and q_d are the spin 1/2 Dirac fields of the quarks, $D_\mu q_i = (\partial_\mu - \imath A_\mu)q_i$ is the covariant derivative. Under a gauge transformation the fields transform in the following way ($\Omega(x) \in SU(3)$)

$$\begin{aligned} q_i &\rightarrow \Omega(x)q_i \\ A_\mu &\rightarrow \Omega(x)A_\mu\Omega^+(x) + \imath\Omega(x)\partial_\mu\Omega^+(x) \end{aligned} \tag{2.34}$$

The fact that the theory is defined in terms of quarks and gluons, whereas we only see the strong interaction through residual forces between hadrons (particles made of two or three quarks) must make us suspect that the ground state of the theory is made of big fluctuations of the elementary fields. The only dimensionful parameters that appear in the QCD Lagrangian are the masses of the quarks, but at least for the lightest quarks, the QCD dynamics seems not to note the value of these masses. For example the mass of the proton will change by less than 5% if we set the quark masses to zero. The QCD scale $\Lambda_{QCD} \approx 150$ MeV is very high to be a consequence of the quark masses, this scale is a consequence of the renormalization process, that needs a scale, via a mechanism called *dimensional transmutation* [11]. The limit in which the quark masses are set to zero is a very good approximation of real QCD, called the *chiral limit*, and we will soon see its implications

To understand what we mean by chiral limit we will analyse the symmetries (other than the gauge symmetry) that our QCD Lagrangian has. If the masses of the quarks u and d were the same we would have a global $U(2)$ symmetry ($U \in SU(2)$)

$$q_i \rightarrow U_{ij}q_j \tag{2.35}$$

this means that we are free to choose what we call an u quark or a d quark. Particular consequences of this symmetry is that if there exist a particle made of the quarks uuu, there should be another particle made of uud, and also one of udd, etc... of the same mass. If we see spin 1/2 particles, there are only two possible particles made of the u and d quarks: the proton ($p = (uud)$) and the neutron ($n = (udd)$), and both of them have approximately the same mass ($m_p = 938.3$ MeV and $m_n = 939.6$ MeV). The small difference can be attributed to the mass difference between the u and d quarks. If we look for spin 3/2 particles we have the four possibilities ((uuu), (uud), (udd), (ddd)), and these are the Δ family, all of them with a mass around 1.232 MeV. This global $U(2)$ symmetry is nothing but a generalisation of the isospin symmetry well known in nuclear physics, and is well confirmed experimentally.

Furthermore, in the limit in which the quark masses are zero, the previous symmetry is amplified to a $U(2)\times U(2)$ symmetry, where besides the previous transformations, we can make transformations of the form

$$q \rightarrow Uq \tag{2.36}$$

where the matrix U acts both in the colour and flavour indices

$$U = e^{\imath\gamma_5(\alpha_a\tau_a+\alpha_0)} \tag{2.37}$$

i.e. we can rotate the left and right components of the quarks in an independent way. For a hadron, these transformations change the parity of the state, since it is a symmetry, hadrons must appear in parity pairs of equal mass. But this is not the case. A fast look through the particle data book web site[3] shows that the parity partner of the proton is the N^0, whose mass is $m_N \approx 1535$ MeV, meanwhile the mass of the proton is $m_p = 938.3$ MeV. This huge difference cannot be produced by the small quark masses[4], so the only way out is that the chiral symmetry is spontaneously broken, and so our ground state is not invariant under these kind of transformations. But as you can imagine, in perturbation theory you cannot break this symmetry, so non perturbative processes must be responsible of the spontaneous chiral symmetry breaking. Later we will analyse this in detail, but for the moment let us examine the consequences of the spontaneous breaking of the chiral symmetry.

2.4 Instantons and the solution of the $U(1)$ problem

Goldstone theorem tells us that if a symmetry is spontaneously broken we must have scalar particles of zero mass, invariant under all the symmetries, but the symmetry that we are breaking. In fact we must have one particle for each generator of the broken symmetry group. In the case of chiral symmetry, the symmetry group is reduced from $U(2) \times U(2)$ to $U(2)$, so we expect 4 particles of zero mass, with all the quantum numbers zero and negative parity.

If we go again to the particle data group web, we find three mesons, that are clearly lighter than the others: The pions π^+, π^- and π^0. Their mass is about 150 MeV, lighter than any other. This mass can be rigorously explained by the value of the quark masses, that makes the chiral symmetry not exact. The candidate for the fourth Goldstone boson, is the η particle, but it's mass $m_\eta \approx 550$ MeV seems very high. It is not only strange that if the chiral symmetry is broken by the same quark masses for the pions and the eta particles their masses are so different, but completely false. It can be proved [12, 13] that if we want the η be the missing Goldstone boson, it's mass cannot be higher than $\sqrt{3}\, m_\pi$. Really the η particle has nothing to do with the chiral symmetry.

This problem of the missing Goldstone boson, is usually called the $U(1)$ problem, because it is the Goldstone boson of this subgroup of the broken $U(2) = SU(2) \times U(1)$ symmetry that we cannot find in the particle spectrum.

The solution to this problem was first pointed out by 't Hooft [14]. We have seen that the action is invariant under the $U(1)$ chiral symmetry group, so the corresponding current

$$J_\mu^5 = \overline{\psi}\gamma_\mu\gamma_5\psi \tag{2.38}$$

is conserved

$$\partial_\mu J_\mu^5 = 0 \tag{2.39}$$

but now we want to look if this statement survives the quantisation process. Classical symmetries that do not survive the quantisation process are known as anomalous symmetries, and from a quantum point of view they are not symmetries at all. One way of seeing how this can happen, is that to properly define a quantum theory we have to renormalize it, and to do that we have to introduce a regularisation method. For example, using the Pauli-Villards regularisation, we introduce imaginary particles of mass M to absorb the divergences, later the mass of these imaginary particles is set to infinity, thus decoupling of the theory and leaving us only with the original particle spectrum. The problem is that introducing massive particles

[3] `http://pdg.lbl.gov/`

[4] There are rigorous theorems about that. See [12] for details.

destroys the chiral symmetry, hence this symmetry would not survive the quantisation process. It is important to note that this is not an effect of a particular regularisation method, if for example we use dimensional regularisation we have difficulties in defining chirality in an arbitrary number of dimensions, it is a problem of the quantum theory in itself, and will appear in any regularisation method that we use. Probably the simplest way of seeing this is using the path integral formalism of quantum field theory: The action is only half of the quantum theory, the other part is the integration measure. For a symmetry to be a quantum symmetry it must leave invariant both the action (that is the same as being a symmetry at the classical level), and the integration measure. It can be proved that chiral symmetry does not leave the measure invariant, so it is not a symmetry at the quantum level. In fact it can be proved that in terms of operators acting on the Hilbert space

$$\partial^\mu \hat{J}^5_\mu \propto \hat{F}^a_{\mu\nu} \tilde{\hat{F}}^{a\mu\nu} \tag{2.40}$$

This is the non abelian version of the Adler-Bell-Jackiw anomaly. Now we will briefly leave the solution to the $U(1)$ problem to study instantons in gauge theories.

As we have said, instantons are classical solutions to the Euclidean equations of motion. They represent tunnelling between different vacua in Minkowski space-time. For a gauge theory, the Euclidean action is given by

$$S_{\text{euc}} = \frac{1}{4g^2} \int d^4x \, F^a_{\alpha\beta} F^a_{\alpha\beta} \tag{2.41}$$

we will solve the equations of motion with the help of the following trick (originally discovered in [15]): consider the trivial inequality

$$\int d^4x \left(F^a_{\alpha\beta} - \tilde{F}^a_{\alpha\beta} \right)^2 \geq 0 \tag{2.42}$$

where $\tilde{F}^a_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\delta\gamma} F^a_{\delta\gamma}$ is the dual field strength. On the other hand, using that

$$\int d^4x \, F^a_{\alpha\beta} F^a_{\alpha\beta} = \int d^4x \, \tilde{F}^a_{\alpha\beta} \tilde{F}^a_{\alpha\beta} \tag{2.43}$$

we can write

$$\int d^4x \left(F^a_{\alpha\beta} \pm \tilde{F}^a_{\alpha\beta} \right)^2 = \int d^4x \left(2F^a_{\alpha\beta} F^a_{\alpha\beta} \pm 2F^a_{\alpha\beta} \tilde{F}^a_{\alpha\beta} \right) = 8g^2 S_{\text{euc}} \pm 2 \int d^4x \, F^a_{\alpha\beta} \tilde{F}^a_{\alpha\beta} \tag{2.44}$$

The last integral is a very important mathematical quantity, related with the so called Pontryagin index or topological charge whose value is always an integer, and invariant under continuous transformations of the gauge fields

$$Q = \frac{1}{32\pi^2} \int d^4x \, F^a_{\alpha\beta} \tilde{F}^a_{\alpha\beta} \tag{2.45}$$

If this last integer is positive, we can choose the $-$ sign in equation (2.44), and obtain

$$S_{\text{euc}} \geq \frac{8\pi}{g^2} Q \tag{2.46}$$

if $Q < 0$ we choose the $+$ sign, and obtain

$$S_{\text{euc}} \geq -\frac{8\pi}{g^2} Q \tag{2.47}$$

In any case, we obtain a non trivial bound to the Euclidean action, that is saturated iif

$$F^a_{\alpha\beta} = \pm \tilde{F}^a_{\alpha\beta} \tag{2.48}$$

the sign depends on the sign of Q. So the work of finding the solutions to the Euclidean equations of motion, that in principle are second order equations in A^a_α, can be reduced to finding (anti-)self dual configurations. So instantons are nothing but self dual Yang-Mills fields A^a_α, usually an anti-self dual configuration is called anti-instanton. One can explicitly construct this self dual configurations

$$A^a_\alpha(x) = \frac{2\eta^a_{\alpha\beta} x_\beta \rho^2}{x^2 + \rho^2} \tag{2.49}$$

where ρ is an arbitrary constant, and the self dual tensor $\eta^a_{\alpha\beta}$ is one of the 't Hooft symbols [14] (for an explicit definition of these symbols see appendix A). The field strength corresponding to this gauge field is

$$F^a_{\alpha\beta} = \tilde{F}^a_{\alpha\beta} = -\frac{4\eta^a_{\alpha\beta}\rho^2}{(x^2 + \rho^2)^2} \tag{2.50}$$

We can see that this solution represent a localised object in 4D Euclidean space time, with a size given by the value of the parameter ρ, hence the name "instanton". More general self dual configurations can be constructed. In essence they are a "superposition" of instantons. These multi instanton configurations can be constructed, in principle, via the ADHM construction [16, 17], that reduces the construction of all the self dual configurations to the much more easy problem of solving a system of algebraic equations.

Now we can see the solution to the $U(1)$ problem. The chiral conserved charge, defined as

$$Q^5 = \int d^3x J^5_0 \tag{2.51}$$

change by two units by the presence of an instanton configuration. We can interpret this as this charge changing under a tunnelling from one minima of the YM vacua to other. Now it is clear that Q^5 is not a conserved quantity: non perturbative effects (instantons in this case), change its value. Instantons have an *effective* interaction with fermions, and change their helicity under this interaction as we can see in the figure (2.5).

2.5 QCD vacuum

In the previous section we have seen one of the two main ingredients of the QCD vacuum: The existence of a global chiral $SU(2)$ symmetry, that is spontaneously broken, with the pions playing the role of Goldstone bosons. The other is colour confinement: all the particles seen in nature are "white" (they live in the 0 representation of the $SU(3)$ gauge symmetry group).

Spontaneous chiral symmetry breaking has a high impact on the properties on the matter that makes our world. It makes the pions light, and hadrons much heavy, and makes possible the formation of the nuclei by a force between hadrons mediated at low energies by pions.

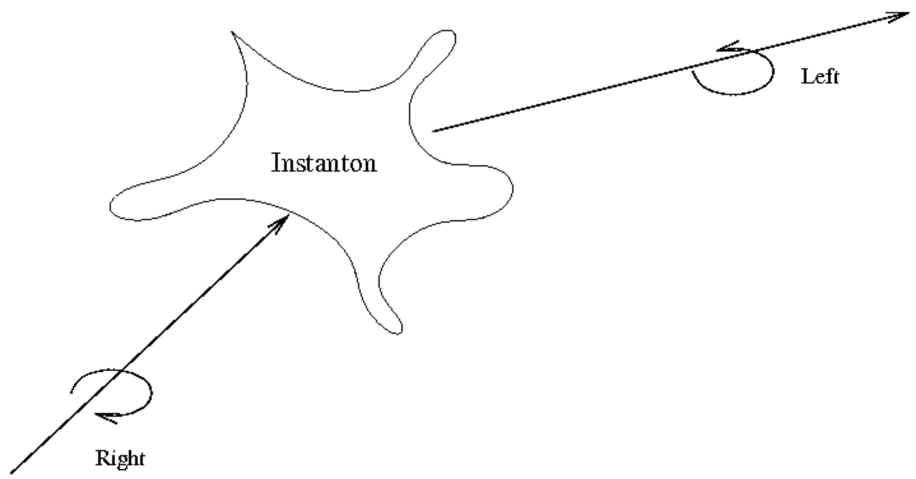


Figure 2.5: The "interaction" between a fermion and an instanton change the helicity of the latter.

Confinement explains why we cannot observe isolated quarks as a matter of principle, and why we only feel an, in principle, large range interaction, like the strong interaction, by the residual forces between non coloured objects made of coloured components.

Any model trying to describe the QCD vacuum must have these two essential ingredients: Spontaneous chiral symmetry breaking and confinement. Although, as we will see later, an instanton dominated vacuum breaks the chiral symmetry, they do not seem to confine, and then this model of the vacuum cannot be the end of the story. There are other large fluctuations of the gauge fields that maybe can produce confinement.

2.5.1 Spontaneous chiral symmetry breaking

It is easy to see that under a chiral transformation the operator $\overline{q}q$ changes by a multiplicative factor. Thus, the spontaneous chiral symmetry breaking is characterised by a non zero quark condensate. In fact it's experimental value is

$$|<\overline{q}q>| = (250 \text{ MeV})^3 \tag{2.52}$$

and it is the main parameter that one has to explain after proposing a model of the QCD vacuum.

To compute this quantity one usually makes use of the Banks-Casher relation [18], that relates the value of the quark condensate with the density of zero modes of the Dirac operator $\not{D} = \gamma^\mu (\partial_\mu - \imath A_\mu)$. Calling this density $\rho(\lambda)$, we have

$$<\overline{q}q> \propto \pi\rho(0) \tag{2.53}$$

So when one has an specific model of the QCD vacua one usually uses this relation to compute the value of the quark condensate, and compare it with the experimental value. If both agree we can say that we have explained the spontaneous chiral symmetry breaking.

On the other hand, the difference between the number of positive chirality and negative chirality zero modes of the Dirac equation have a relation with the topological charge Q of the background gauge field $A_\mu(x)$. In fact if n_+ is the number of positive chirality zero modes, and n_- is the number of negative chirality ones

$$Q = n_+ - n_- \tag{2.54}$$

This key observation is a particular case of the general Atiyah-Singer index theorem [19]. In particular in the presence of a instanton we have exactly one zero mode with positive chirality, and in the presence of an anti-instanton one has a zero mode of negative chirality.

If we have one instanton and one anti-instanton infinitely separated, we will have two zero modes with opposite chirality. When the instanton and the anti-instanton are not infinitely separated, the two would-be zero modes converts in two eigenfunctions with eigenvalue $\pm\lambda$, that can be computed as the overlap integral between the wave function of the Dirac operator in the instanton background and the one in the anti-instanton background. In general if we have an ensemble of many instantons and anti-instantons, we expect that the would-be zero modes of each individual instanton produce an spectral density of the Dirac operator. In the particular case of a dilute gas of instantons, the spectral density can be computed [20], and also the chiral condensate, obtaining

$$| < \overline{q}q > | \approx (253 \text{ MeV})^3 \tag{2.55}$$

very close to the measured value. It is important to note that the fact that an instanton dominated vacuum breaks the chiral symmetry, and hence gives an explanation to the low energy hadron spectrum does not mean that this is *the way* that QCD uses to break the chiral symmetry. In the explanation that we have followed, the key ingredient for the success of the instanton model was the generation of a non zero density of zero modes of the Dirac operator, and it is worth to say that there are many semi-classical configurations that also can generate a non zero density of zero modes of the Dirac operator. In fact it can be argued that via the Atiyah-Singer index theorem, any semi-classical configuration that carries a topological charge will do the job.

2.5.2 Confinement

The other main characteristic of the QCD vacuum is colour confinement: all the hadrons that are seen in nature are colour singlets, made of quarks and gluons that are *not* colour singlets. If you try to "ionise" a hadron (i.e. to break the hadron in its coloured components), you always get more colour singlet hadrons, no matter how much energy you use to "break" the hadron. A picture of what happens when you try to separate two quarks can be seen in the figure (2.6): the energy stored in the colour field that binds the two quarks increases with the distance, making the system unstable under the production of a new pair of particles. Hence if you try to separate two quarks, to obtain an isolated quark, you will always obtain new hadrons. To examine in detail this property, it is convenient to explore the previous situation in the limit of infinite quark mass. In this limit the production of a pair of quark-antiquark is impossible, and hence we can study the potential energy as a function of the separation between the quarks. This procedure gives rise to the static quark potential, whose form is analytically unknown, but some general properties are known: the force between quarks is always attractive, but it does not increase with the distance. This means that the potential cannot rise faster than linearly with the distance between quarks. One can convince oneself that in fact the quark potential is asymptotically linear

$$\lim_{R\to\infty} V(R) = \sigma R \tag{2.56}$$

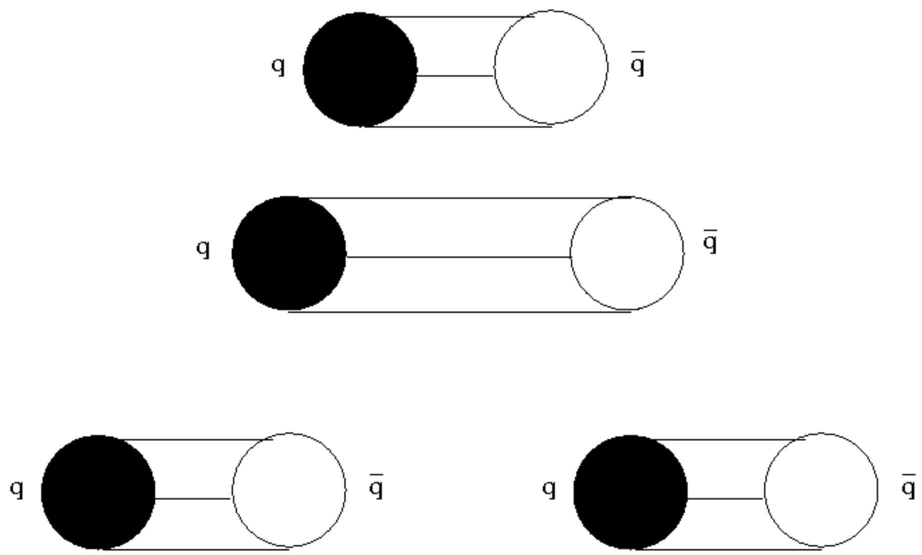


Figure 2.6: When you separate two quarks the energy of the colour field increases, making the pair of quarks unstable under the creation of another pair of quark-antiquark

where σ is known as the string tension, whose specific value can be obtained by the low-lying meson phenomenology

$$\sigma \approx 0.9\,\mathrm{GeV/fm} \tag{2.57}$$

This is other of the phenomenological parameters that a QCD vacuum model should explain. Although, as we have seen, instantons explains the $U(1)$ problem, and a instanton dominated vacuum can produce a spontaneous breaking of the chiral symmetry, it does not produce a growing potential at large distances [21].

There are several models for the QCD vacuum that try to explain confinement, but the most popular ones are based on the idea that the functional integral of QCD is dominated by some field configurations (like instantons, although as we have said this model does not seem to confine) that make the potential grow at large distances. There are several options for these field configurations: monopoles, centre vortices, merons, calorons, etc... A good review on this subject is [22], but they share the common properties that they are "based" on semi-classical configurations. As we have mentioned before, a deep understanding of these classical configurations is necessary to study if they confine or not. Later we will explore some of these semi-classical configurations.

2.6 Gauge fields in the Torus

There are several reasons to study gauge fields in the Torus. First, configurations in the Torus can be interpreted as periodic configurations in $\mathbb{R}^n$. The torus is the natural space if we are interested in studying properties of a system with a constant density of objects.

Second, in the limit in which the length of the torus sides is taken very large, we again recover configurations in $\mathbb{R}^n$. In this sense, configurations in $\mathbb{R}^n$ are special cases of configurations in the torus. In fact, in the following chapter we will exploit this way of proceeding to obtain quantities of $\mathbb{R}^n$ from our solutions in the torus, and we will show that this is a good method to obtain these values, at least competitive in precision with other methods specific for $\mathbb{R}^n$.

Third, the non trivial topology of the torus can be used to characterise some non perturbative properties of the theory, including confinement. In fact this was the original motivation of 't Hooft [23, 24] to study gauge fields in the torus: He wanted to characterise confinement in terms of the non trivial topology of the base space.

Fourth, the size of the torus can be used as a parameter which can be used to interpolate between the perturbative and non perturbative regions of the theory. This can be used to approximately compute some non perturbative quantities, and can give some insight in the structure of the Yang-Mill vacuum [25, 26].

More information about this topic can be found in [27], and the references therein.

In what follows we will be working in four dimensional Euclidean space time, and with an $SU(2)$ gauge theory. Although some ideas are more general, and independent on the dimension of space time, and will be used later for an abelian gauge theory in two dimensions.

We will work in a torus of length sizes l_α, and the vectors $\vec{l}_\alpha$ are defined as follows

$$\begin{aligned}
\vec{l}_1 &= (l_1, 0, 0, 0) \\
\vec{l}_2 &= (0, l_2, 0, 0) \\
\vec{l}_3 &= (0, 0, l_3, 0) \\
\vec{l}_4 &= (0, 0, 0, l_4)
\end{aligned} \tag{2.58}$$

we will use a flat metric on the torus, and hence the volume of the torus is given by

$$\mathcal{V} = l_1 l_2 l_3 l_4 \tag{2.59}$$

The key idea to understand gauge fields in the torus, is that since the point x and the point $x + \vec{l}_\alpha$ are the same the gauge connection $A_\alpha(x)$ and $A_\alpha(x + \vec{l}_\beta)$ must be physically the same connection. This does not necessary means that they are equal, it is enough for one of them to be a gauge transform of the other

$$A_\alpha(x + \vec{l}_\beta) = [\Omega_\beta(x)] A_\alpha(x) \tag{2.60}$$

the $SU(2)$ elements $\Omega_\alpha(x)$ are called transition or twist matrices. The consistency of this boundary condition requires that you can go to the point $(x + \vec{l}_1 + \vec{l}_2)$ first doing the gauge transformation along the $\vec{l}_1$ direction and after along the $\vec{l}_2$ direction, or in the other order, and obtain the same result. This requires

$$\Omega_\alpha(x + \vec{l}_\beta)\Omega_\beta(x) = z_{\alpha\beta}\Omega_\beta(x + \vec{l}_\alpha)\Omega_\alpha(x) \tag{2.61}$$

where $z_{\alpha\beta}$ is an element of the centre of $SU(2)$, that is Z_2. So we can express it by

$$z_{\alpha\beta} = \exp\left(2\pi\imath \frac{n_{\alpha\beta}}{2}\right) \tag{2.62}$$

where $n_{\alpha\beta}$ is an antisymmetric tensor of integers modulo 2 (so its only possible values are 0 and 1), usually called the twist tensor. Being an antisymmetric tensor in 4 dimensions, it has 6 independent coefficients, that can be expressed in terms of two three vectors k_i and m_i defined by

$$\begin{aligned}
n_{0i} &= k_i \\
n_{ij} &= \varepsilon_{ijk} m_k
\end{aligned} \tag{2.63}$$

These boundary conditions are usually called twisted boundary conditions. The particular case in which $n_{\alpha\beta} = 0$ is usually referred as non twisted boundary conditions, and since under a gauge transformation $U(x) \in SU(2)$ the twist matrices changes as follows

$$\Omega_\alpha(x) \longrightarrow U(x + \vec{l}_\alpha)\Omega_\alpha(x)U^+(x) \tag{2.64}$$

but the twist tensor does not change. The particular case in which

$$\kappa(n_{\alpha\beta}) \equiv k_i m_i = 0 \pmod 2 \tag{2.65}$$

are called orthogonal twisted boundary conditions.

It is easy to find twist matrices for a particular twist tensor: the commuting matrices

$$\Omega_\alpha(x) = \exp\left(\frac{\imath\pi}{2}\sum_\beta n_{\alpha\beta}\frac{x_\beta}{l_\beta}\tau_3\right) \tag{2.66}$$

do the job. Of course these "abelian twist matrices" are not the only option, in particular any gauge transform of these twist matrices will also do the job, as we have seen, and

$$\Omega'_\alpha(x) = z_\alpha \Omega_\alpha(x) \tag{2.67}$$

where z_α is an element of Z_2, the centre of $SU(2)$, will also work. Another option for the twist matrices is to choose them constant but non-commuting. These are usually referred as twist eaters. Much more information about this topic can be found in [27].

The self duality equations, being local equations, are the same in the torus than in $\mathbb{R}^4$, and we still have that the action is bounded from below by the value of the topological charge. This bound is saturated if and only if the gauge field is (anti) self dual. The topological charge can be expressed in terms of the twist matrices (for the details see [27])

$$\begin{aligned} Q \quad = \quad & -\frac{1}{24\pi^2}\varepsilon_{\alpha\beta\gamma\delta}\int_{\mathrm{T}^4} d\sigma_\alpha \mathrm{Tr}\left[\Omega^+_\alpha(\partial_\beta\Omega_\alpha)\Omega^+_\alpha(\partial_\gamma\Omega_\alpha)\Omega^+_\alpha(\partial_\delta\Omega_\alpha)\right] - \\ & -\frac{1}{24\pi^2}\varepsilon_{\alpha\beta\gamma\delta}\int_{\mathrm{T}^4} d\sigma_\alpha \mathrm{Tr}\left[(\Delta_\gamma \overline{A}^{(\alpha)}_\beta + \overline{A}^{(\alpha)}_\beta)\Omega_\gamma\partial_\delta\Omega^+_\gamma\right] \end{aligned} \tag{2.68}$$

where $d\sigma_\alpha$ is the integral element over the $x_\alpha = 0$ 3D face of the torus, $\overline{A}^{(\beta)}_\alpha = \imath\Omega^+_\beta\partial_\alpha\Omega_\beta$ and the operator Δ_α acts in the following way

$$\Delta_\alpha f(x) = f(x + \vec{l}_\alpha) - f(x) \tag{2.69}$$

For the special case of the twist matrices that we will use (see equation (2.66)), the topological charge is given by

$$Q = -\frac{\kappa(n_{\alpha\beta})}{2} \tag{2.70}$$

This means that zero action configurations are only possible if we have orthogonal twisted boundary conditions.

Not much is known analytically about self dual configurations in the torus for arbitrary sizes and topological charge, but the existence of self dual configurations in the twisted torus [28]. A generalisation of the ADHM construction, called the Nahm transform [29], that works for fields living on different manifolds does not solve the problem for the case of the torus, but map self dual configurations of topological charge Q of a $SU(N)$ theory in self dual configurations of topological charge $Q' = N/N_0$ of a $SU(N_0 Q)$ gauge theory, being N_0 an integer that depends on the twist [30] (for the particular case of non twisted boundary conditions, $N_0 = 1$). One usually has to turn to numerical methods to construct the self dual configurations in the torus [31].

On the other hand, the knowledge of analytic solutions is a desirable thing. The reason why instanton methods are almost the only known analytical method to get non perturbative observable quantities for QCD, is basically the fact that classical instanton solutions are very well understood. The path that leads to the computation of a quantity like the chiral condensate, and that pass through a quantitative treatment of the instanton ensemble, starts in a good understanding of the classical configurations.

2.6.1 Perturbative construction of self dual configurations in the Torus

In this section we will introduce the results of [32], that we can consider the origin of the Bradlow parameter expansion. In what follows, we will work with the twist tensor

$$n_{\alpha\beta} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{2.71}$$

which is non orthogonal. The topological charge has a value $Q = 1/2$. The generalisation to other twist tensor is straightforward. We will define the Bradlow parameter

$$\epsilon = \frac{l_4 l_3 - l_1 l_2}{\sqrt{\mathcal{V}}} \tag{2.72}$$

The special gauge connection, consistent with our choice of twist matrices (equation (2.66)) given by

$$A_\alpha^{(0)}(x) = -\frac{\pi}{2} n_{\alpha\beta} \frac{x_\beta}{l_\alpha l_\beta} \tau_3 \tag{2.73}$$

gives rise to the constant field strength

$$F_{\alpha\beta}^{(0)} = \pi \frac{n_{\alpha\beta}}{l_\alpha l_\beta} \tau_3 \tag{2.74}$$

It's only non zero components are $F_{34}^{(0)}$ and $F_{12}^{(0)}$, which are of equal magnitude at $\epsilon = 0$, making the solution self dual. So we have found self dual configurations in the four dimensional torus with constant field strength for some torus sizes. These solutions were well known, and the first to find them was 't Hooft [33]. The key idea of the Bradlow parameter expansion is to write a gauge field that is close to the one that gives rise to a constant field strength

$$A_\alpha(x) = A_\alpha^{(0)}(x) + S_\alpha(x)\tau_3 + W_\alpha(x)\tau_+ + \overline{W}_\alpha(x)\tau_- \tag{2.75}$$

where we have decomposed the perturbation around $A_\alpha^{(0)}(x)$ in its colour components. The boundary conditions for the gauge field of equation (2.60), translates into the following conditions for $S_\alpha(x)$ and $W_\alpha(x)$

$$S_\alpha(x + \vec{l}_\beta) = S_\alpha(x) \tag{2.76a}$$

$$W_\alpha(x + \vec{l}_\beta) = \exp\left(\imath\pi n_{\beta\gamma}\frac{x_\gamma}{l_\gamma}\right) W_\alpha(x) \tag{2.76b}$$

Since for $\epsilon = 0$, the self duality equations are solved with the values

$$S_\alpha(x) = W_\alpha(x) = 0 \tag{2.77}$$

it is natural to try to compute both $S_\alpha(x)$ and $W_\alpha(x)$ as a power expansion in the parameter ϵ for the solution to be self dual for an arbitrary value of the torus length (for arbitrary ϵ). Using the notation described in appendix A, we will contract both $S_\alpha(x)$ and $W_\alpha(x)$ with $\sigma_\alpha = (-\imath\tau_i, \mathbb{I})$. We will work in the background field gauge

$$\partial_\alpha A_\alpha(x) - \imath[A^{(0)}_\alpha(x), A_\alpha(x)] = 0 \tag{2.78}$$

and we will demand to the full strength field to be self dual

$$F_{\alpha\beta} = \tilde{F}_{\alpha\beta} \tag{2.79}$$

contracting both $F_{\alpha\beta}$ with $\overline{\sigma}_\alpha\sigma_\beta$ we project out the self dual part of $F_{\alpha\beta}$, and allows us to rewrite the self duality equation as equations for the matrices S and W

$$\overline{D}W = \imath\left(S^+W - W_c^+S\right) \tag{2.80a}$$

$$\overline{\partial}S = \frac{1}{2}\hat{F}^{(0)} + \frac{\imath}{2}\left(W_c^+W_c - W^+W\right) \tag{2.80b}$$

where $D_\alpha = \partial_\alpha + \imath\pi x_\beta n_{\alpha\beta}/(l_\alpha l_\beta)$ and W_c is the charge conjugate defined by

$$W_c = \tau_2 W^* \tau_2 \tag{2.81}$$

and $\hat{F}^{(0)}$ is given by

$$\hat{F}^{(0)} \equiv F^{(0)}_{\alpha\beta}\overline{\sigma}_\alpha\sigma_\beta = 2\pi\imath\frac{\epsilon}{\sqrt{\mathcal{V}}}\tau_3 \tag{2.82}$$

The idea is to solve the self duality equations (2.80) using a power series for the matrices S and W. Equations (2.80) are consistent with the following expansions

$$S = \sum_{k=1}^{\infty} S^{(k)}\epsilon^k \tag{2.83a}$$

$$W = \sqrt{\epsilon}\sum_{k=0}^{\infty} W^{(k)}\epsilon^k \tag{2.83b}$$

Our task now is reduced to solve the self duality equations (2.80) order by order in the series expansion.

Existence and uniqueness of the solutions

The first self duality equation (2.80a) reads to first order in ϵ

$$\overline{D}W^{(1)} = 0 \tag{2.84}$$

As we can see in [34] this equation can be interpreted in terms of creation and annihilation operators acting in the Hilbert space of functions with the periodicity conditions (2.76b), and it's general solution is given by

$$W^{(1)} = \Psi(x)\begin{pmatrix} K^{(1)} & Q^{(1)} \\ 0 & 0 \end{pmatrix} \tag{2.85}$$

where $K^{(1)}$ and $Q^{(1)}$ are two arbitrary complex constants, and $\Psi(x)$ is a function with the same boundary conditions as the field W_α, that can be written in terms of the Jacobi theta functions studied in the appendix B

$$\begin{aligned}\Psi(x) &= \sqrt{\frac{4l_3l_2}{l_4l_1}}\exp\left\{-\frac{\pi}{l_4l_3}(x_3^2-\imath x_3x_4)-\frac{\pi}{l_1l_2}(x_2^2-\imath x_1x_2)\right\}\\ &\times\ \vartheta_3\left(\frac{\pi(x_1+\imath x_2)}{l_1}|\imath\frac{l_2}{l_1}\right)\vartheta_3\left(\frac{\pi(x_4+\imath x_3)}{l_4}|\imath\frac{l_3}{l_4}\right)\end{aligned} \tag{2.86}$$

To higher orders in ϵ, the first self duality equation can be solved in the following way: we write

$$W^{(k)} = DU^{(k)} + \Psi(x)\begin{pmatrix} K^{(k)} & Q^{(k)} \\ 0 & 0\end{pmatrix} \tag{2.87}$$

where $K^{(k)}$ and $Q^{(k)}$ are again arbitrary complex constants and $U^{(k)}$ is a solution of

$$\overline{D}DU^{(k)} = \imath\sum_{l=1}^{k-1}\left(S^{(k-l)}W^{(l)} - W_c^{(l)+}S^{(k-l)}\right) \tag{2.88}$$

Since the operator $\overline{D}D$ is invertible (has no zero modes), this equation gives a unique solution $U^{(k)}$ in terms of $U^{(l)}$ and $S^{(l)}$ for $0 < l < k$. As we have seen, following this procedure we have to fix two complex constant for each order in ϵ.

The other self duality equation (2.80b) reads

$$\bar{\partial}S = \frac{1}{2}\hat{F}^{(0)} + \frac{\imath}{2}\left(W_c^+W_c - W^+W\right) \tag{2.89}$$

where $S(x)$ is a periodic matrix. It is natural to expand both sides of equation (2.80b) in a Fourier series. If we are able to compute the Fourier modes of the r.h.s. of equation (2.80b) order by order, we will be able to compute the Fourier series of $S(x)$, and in fact it is easy to use the results of the appendix B to compute the Fourier series of $|\Psi(x)|^2$. The l.h.s. of equation (2.80b), being the derivative of a quantity, has a vanishing zero Fourier mode, and this fact can be used to fix, order by order, some of the in principle arbitrary constants $K^{(k)}$ and $Q^{(k)}$. To order ϵ^k, the dependence of equation (2.80b) in these parameters is of the following form

$$|\Psi(x)|^2(c_3\tau_3 + c_+\tau_+ + c_-\tau_-) \tag{2.90}$$

where $c_3 = 2\mathrm{Re}(Q^{(k)}Q^{(1)*} - K^{(k)}K^{(1)*})$ and $c_+ = 2(K^{(k)*}Q^{(1)} + K^{(1)*}Q^{(k)})$. Since $|\Psi(x)|^2$ has a non vanishing zero Fourier mode, the complex constants $K^{(k)}$ and $Q^{(k)}$ can be chosen in such a way that the zero Fourier mode of the r.h.s. of equation (2.80b) vanish. Although this procedure ensures that the solution to equation (2.80b) exists order by order, we still have two things to study: first our procedure only fixes three of the four real parameters which enter in $K^{(k)}$ and $Q^{(k)}$, and second the solution for $S(x)$ is not unique because we are free to add a constant matrix to $S(x)$, and this does not change equation (2.80b).

The fact that we have not been able to determine the solution in a unique way is natural, and is associated with the existence of transformations that change one solution into other. These transformations are remaining global gauge transformations of a certain kind, associated with the freedom to multiply W by a constant phase, and the translational invariance, associated with the four parameters that enter in the constant term of $S(x)$. We can fix a unique solution by choosing a particular solution, so we will impose the additional conditions

$$\begin{aligned} \mathrm{Re}(W_{12}(x=0)) &= |W_{12}(x=0))| && (2.91a) \\ \int_{\mathbb{T}^4} d^4x\, S(X) &= 0 && (2.91b) \end{aligned}$$

Summarising, we have proved the existence of a solution to the self duality equations order by order in the parameter ϵ, that measures how far our torus is from the "critical" size for which solutions with constant field strength exist. If we constrain the transformations that change one solution into another, our procedure gives a unique solution to the self duality equations.

First order computation

Here we will exemplify the previous procedure by computing the first order in the expansion. We have seen that to first order W is given by

$$W^{(1)} = \Psi(x) \begin{pmatrix} K^{(1)} & Q^{(1)} \\ 0 & 0 \end{pmatrix} \tag{2.92}$$

where $\Psi(x)$ is a known function. Using equation (2.91a) we have

$$\begin{aligned} K^{(1)} &= 0 && (2.93a) \\ Q^{(1)} &= \frac{\sqrt{2\pi}}{\sqrt[4]{\mathcal{V}}} && (2.93b) \end{aligned}$$

and $W^{(1)}$ is completely determined. The equation for $S(x)$ to first order is given by

$$\overline{\partial} S^{(1)} = \frac{\imath\pi}{\sqrt{\mathcal{V}}}(|\Psi|^2 - 1)\tau_3 \tag{2.94}$$

and is solved writing

$$S^{(1)} = \frac{\imath\pi}{\sqrt{\mathcal{V}}}(\partial h)\tau_3 \tag{2.95}$$

where $h(x)$ is a periodic function, solution of

$$\partial_\alpha \partial_\alpha h = |\Psi|^2 - 1 \tag{2.96}$$

Using the results of the appendix B, we can compute the Fourier modes of $|\Psi(x)|^2$

$$\begin{aligned} \mathcal{F}[|\Psi|^2] &= \exp\left\{ -\frac{\xi(n_1, n_2, l_2/l_1) + \xi(n_4, n_3, l_3/l_4)}{2} \right\} \\ &\times \exp\left\{ \imath\pi \frac{n_1 n_2 + n_3 n_4}{2} \right\} && (2.97) \end{aligned}$$

where

$$\xi(n, m, \tau) = \pi\tau\left(n^2 + \frac{m^2}{\tau}\right) \tag{2.98}$$

and then we can explicitly obtain the Fourier modes of both $h(x)$ and $S(x)$, that completely determine the solution to first order.

Higher order computations, and some comments

We can compute higher orders in the expansion in ϵ, although it is hard to obtain closed analytical expressions. Higher orders can be calculated using a computer. The fact that the Fourier modes of $|\Psi(x)|^2$ decay exponentially fast with the number of modes, means that we can truncate the Fourier expressions of $S(x)$ and obtain a very accurate approximation to the function.

The possibility of computing more orders in the series with the help of a computer is out of any doubt, but the important thing is that having an analytical way of computing the self dual configurations on the torus, and having a simple analytical expression for an approximate self dual configuration, as the first order approximation found above is, open doors to explore more complex questions that are impossible to investigate without the knowledge of the shape of the configurations. For example in the same work that leads to the first order computation of the self dual configurations in the torus (reference [32]), the Nahm transform of this first order approximate solution has been shown to be also the first order of the same expansion in the dual torus. This part of the investigation would be completely impossible without the knowledge of the exact analytical form of the solution, or at least much more difficult.

As we will see in the next section, and in the rest of the work, the idea of using a known configuration only valid (i.e. solution of the appropriate equations) for some torus size, and computing the solutions valid for all torus sizes as a perturbation series is possible in a large class of theories, and not only for the case of pure Yang Mills theories.

2.7 Other non perturbative contributions

Instantons, and the quantum interpretation that we make of these solutions as tunnelling effects are not the only non perturbative contributions in quantum theory, although they are probably the ones that we best understand. Classical solutions in Minkowski space time with finite energy, and localised in space, can also lead to non perturbative contributions to physical quantities. These kind of solutions usually have a non dispersive energy density, so they travel undistorted in shape with uniform velocity. They are generically called “solitons”, although strictly speaking we should call them “solitary waves”, and reserve the name solitons to the objects whose shape and structure remains unchanged even after a collision, in the physicist literature is customary to call these kind of solutions solitons, and we will follow the trend here.

Although solitons are solutions to the classical theory, and we are interested in quantum theory, these soliton solutions turn out to be extended particle states in the quantum theory. It may not seem very surprising, but the procedure to establish this correspondence were developed only in the mid seventies by different people. In fact from each classical soliton solutions arise a tower of quantum states, that are computed quantising the fluctuations around the soliton. The properties of the quantum soliton particle, like it’s mass, can be obtained in a systematic way. There are important difficulties in this process like how to treat the zero modes associated with symmetry transformations that change one solution into another, and that naively gives an infinite contribution to the energy, or how to renormalize the ultraviolet divergences. In fact one can think of the usual perturbation theory of quantum field theory, as a particular case of these techniques, where one has used the trivial vacuum solutions as the classical “soliton” solution to start the quantisation. The thing is complicated in itself, but something is clear, to investigate the properties of the quantum solitons states, we need to know either in exact analytical form (this is the most desirable thing), or at least in good approximation the classical solutions and it's stability properties.

Typically these soliton solutions are classified by some topological number associated with the behaviour of the fields at the spatial infinity, like the topological charge in the case of instantons. Being a topological quantity it cannot change by any continuous deformation of the fields, and in particular cannot be changed by the temporal evolution. This means that this topological number is a conserved quantity, but very different to the other conserved quantities that usually arise in field theory, via the Noether theorem. This topological conserved quantity become a conserved quantum number, characterising the state in the quantum theory. This fact makes the topological analysis of the theory a first step towards the understanding of the properties of the classical solutions, and hence of its corresponding quantum states.

In this section we will briefly review the most important different kind of solitons, the steps are always the same: we begin with a topological analysis of the configurations, and then some comments about the existence of analytical solutions follow. A short review on the important areas of physics where they are relevant will be discussed in each section.

2.7.1 Monopoles

Monopoles have a long history in theoretical physics, beginning with Dirac [35, 36] who was the first to study them. The idea is very simple, we postulate the existence of a particle with a radial magnetic field, the problem is that this kind of magnetic field is in flagrant contradiction with one of the Maxwell equations

$$\mathrm{div} \cdot \vec{B} = 0 \tag{2.99}$$

We will not tell here the story of how to make both the Maxwell equation and the existence of the magnetic monopoles compatible, but simply say that this compatibility makes inevitable the singularity of the gauge field at the origin, that in the end requires the quantisation of the electric charge. The surprising effect that the existence of a single monopole in our world can quantise the electric charges of all particles has made people look for monopoles from both the theoretical and the experimental point of view. The truth is that up to the present date no monopole particle has been detected.

But it was the discovery by 't Hooft [37] and Polyakov [38] in 1974 that monopole solutions are a generic feature of certain gauge theories what revived the interest in this particles more than 40 years after the first work by Dirac. These monopoles are extended non perturbative soliton solutions, and have an important role in cosmology and grand unified theories: the absence of observed monopoles has ruled out some theories of grand unification, or some cosmological scenarios, and, as a matter of fact, the low density of monopoles in our universe is used as an standard argument to support the inflationary paradigm.

We will work out a concrete example, and use an $SU(2)$ non abelian gauge theory coupled with a field living in the adjoint representation of $SU(2)$

$$\phi = \phi^a \frac{\tau_a}{2} \tag{2.100}$$

The Lagrangian of our theory is

$$\mathcal{L} = -\frac{1}{4}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + \frac{1}{2}\mathrm{Tr}\left[(D_\mu\phi)\,(D^\mu\phi)\right] - \frac{\lambda}{8}\left(\mathrm{Tr}(\phi^+\phi) - 1\right)^2 \tag{2.101}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \imath[A_\mu, A_\nu]$ is the field strength, and the covariant derivative acts over our field in the adjoint representation in the usual way

$$D_\mu \phi = \partial_\mu \phi - \imath[A_\mu, \phi] \tag{2.102}$$

Now we are going to classify the finite energy solutions of this model. For the energy to be finite, the energy density must vanish at spatial infinity, so the potential energy density must vanish at infinity

$$\mathrm{Tr}(\phi^+ \phi) = 1 \tag{2.103}$$

this fixes the value of the Higgs field at infinity to some value, we can choose for example

$$\phi(\infty) = \tau_3/2 \tag{2.104}$$

but this is not the only possibility, in general we can use a general gauge transform of this, that depends on the coordinates θ and φ that parametrise the different directions that we can use to approach the infinity

$$\phi = U^+(\theta, \varphi)\frac{\tau_3}{2}U(\theta, \varphi) \tag{2.105}$$

where $U \in SU(2)/U(1)$. The reason why $U(\theta, \varphi)$ belongs to the subgroup $SU(2)/U(1)$ instead of the full group $SU(2)$ is because fixing the value of the Higgs field at infinity (equation (2.104)), does not break the $SU(2)$ group completely, we still have the freedom to multiply the matrices U by a factor

$$W = e^{\imath\alpha\frac{\tau_3}{2}} \tag{2.106}$$

without changing the value of ϕ. We can see our Higgs field as a function, that maps points of the spatial infinity with coordinates θ and φ, to a matrix $U(\theta, \varphi) \in SU(2)/U(1)$. Since the spatial infinity is the two sphere S^2 we are interested in the maps

$$\phi : S^2 \longrightarrow SU(2)/U(1) \tag{2.107}$$

it can be argued that since $SU(2) \sim S^3$ is parametrised by three angles, making the quotient with $U(1)$ will remove one of these angles, so that $SU(2)/U(1)$ depends only on two angles, just like the two sphere S^2, so we are interested in the mappings from the two sphere in the two sphere, that are classified by an integer, the winding number. Mathematically speaking our previous considerations are not rigorous, but since we are interested in the second homotopy group of $SU(2)/U(1)$, we can use the standard mathematical tools of topology to compute

$$\pi_2(SU(2)/U(1)) \sim \pi_1(U(1)) \sim \mathbb{Z} \tag{2.108}$$

obtaining the same result. Now we know that finite energy configurations are classified by an integer, the topological charge, but the asymptotic form of the gauge field is also given by the finite energy condition. As $|\vec{x}| \to \infty$, the term $D_i\phi$ must also vanish, but the term $\partial_i\phi$ at infinity has a term $\partial_i\phi \sim 1/r$, that has to be cancelled in the covariant derivative with a term $A_i \sim 1/r$, and this implies that the magnetic field that corresponds to the unbroken $U(1)$ gauge symmetry (that has to be identified with usual electromagnetism), asymptotically goes as $B \sim 1/r^2$, just as we expect from a monopole. The magnetic charge of the monopole is given by the integer that characterises the map, and as we have said is a conserved number that characterises the state. We can obtain its explicit form (see for example [39]) in terms of the form of the Higgs field at infinity (equation (2.105))

$$m = \int_{S^2} \vec{B} \cdot d\vec{\sigma} = \frac{1}{8\pi} \int_{S^2} \varepsilon^{ijk} \varepsilon^{abc} \phi^a \partial_j \phi^b \partial_k \phi^c d\sigma_i \tag{2.109}$$

In the limit in which $\lambda = 0$, we can write a lower bound for the energy, similar to the bound of the Euclidean action that we found studying instantons. The energy can be written as

$$\begin{aligned} E &= \frac{1}{2} \int \frac{1}{2} F^a_{ij} F^a_{ij} + D_k \phi^a D_k \phi^a \, d^3x \\ &= \frac{1}{4} \int \left(F^a_{ij} - \varepsilon_{ijk} D_k \phi^a \right)^2 \, d^3x + \frac{1}{2} \int \varepsilon_{ijk} F^a_{ij} D_k \phi^a \, d^3x \end{aligned} \tag{2.110}$$

and the last integral can be related with the topological charge via

$$\int \varepsilon_{ijk} F^a_{ij} D_k \phi^a \, d^3x = \int \partial_k \left(\varepsilon_{ijk} F^a_{ij} \phi^a \right) \, d^3x = \int \varepsilon_{ijk} F^a_{ij} \phi^a \, d\sigma_k = \int B_k \, d\sigma_k = 8\pi m \tag{2.111}$$

so the energy can be written as

$$E = \frac{1}{4} \int \left(F^a_{ij} - \varepsilon_{ijk} D_k \phi^a \right)^2 \, d^3x + 4\pi m \tag{2.112}$$

and we have a lower bound to the energy

$$E \geq 4\pi m \tag{2.113}$$

and the bound is saturated iif

$$F^a_{ij} = \varepsilon_{ijk} D_k \phi^a \tag{2.114}$$

that are usually known as Bogomolny equations. This Bogomolny equation is closely related with the self duality equations in Euclidean space time that defines instanton solutions. Monopoles of this model, with $\lambda = 0$, are usually called BPS monopoles, and as we will comment later, they are of central importance in supersymmetric theories. There exist an exact solution to the Bogomolny equations with spherical symmetry, found by Prasad and Sommerfield [40].

2.7.2 Vortices

Vortices are the main ingredient of the rest of the thesis, so we are not going to present here the details of the solutions, this will be covered later, but only sketch the basics. The prototype model to study vortex solutions is the abelian Higgs model, that couples an abelian gauge field with a charged scalar field. Without entering in the details, that will be covered later, the Lagrangian of this theory can be written as

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} |D_\mu \phi|^2 - \frac{\lambda}{8} (|\phi|^2 - 1)^2 \tag{2.115}$$

If we classify the configurations with finite energy, independent of the coordinate x_3, we can see that they are classified by a topological number, that can be interpreted as the total magnetic flux that crosses the $x_1 - x_2$ plane. When these configurations are seen in ordinary $3D$ space

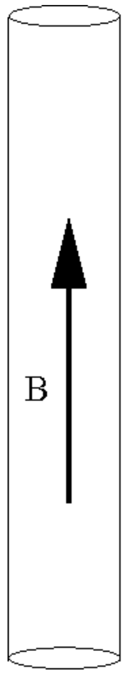


Figure 2.7: A vortex string, carrying magnetic flux through its core.

they are string like configurations that carry a magnetic flux at their core (see figure 2.7). The equations of motion for this model, can again be reduced to a set of first order equations if $\lambda = 1$ by finding an appropiate bound to the energy, these first order equations play the role of the self duality equations in the case of instantons, or of the Bogomolny equations in the case of the monopole solutions

These string like configurations appear in very different areas of physics. In condensed matter systems, these vortex like structures are magnetic flux tubes in superconductors and charged excitations in quantum hall fluids.

In cosmology, these kind of solitons give rise to cosmic strings. Although the original motivation for the study of cosmic strings in the eighties was a way of generating the primordial density perturbations from which the anisotropies of our universe like galaxies and clusters of galaxies arise, now the enormous data coming from COBE, and more recently from WMAP, confirm that topological defects can contribute to the cosmic background anisotropies but they *are not* it's primary source. Nowadays it is the general predictions of string theory, M theory and supersymmetric GUT theories that cosmic strings should exist, together with the possibility of observing these defects through a lensing effect what have revived the interest of the community in these objects. From this point of view, it is both the qualitative and quantitative understanding of the collision of cosmic strings, what is most important. A low energy approximation to these collisions will be the subject of the chapters 6 and 7.

Until this point we have not said anything about the interaction of solitons, although it is one of its most interesting problems, in the case of vortices and its applications in cosmology, this is *the key* problem to solve.

2.8 Solitons and supersymmetry

This picture of classical soliton solutions as quantum particle states that give rise to non perturbative contributions to quantum processes, is still true when one deals with supersymmetric field theories, but in this case we have another motivation to understand these soliton solutions. The key idea is duality, but before exploring this idea, we will make a short comment.

Some values of the couplings that were necessary in the previous sections to simplify the equations of motion, like the special value $\lambda = 0$ (or vanishing potential) for the monopole, or the Bogomolny limit ($\lambda = 1$) for vortices are seen as values put by hand in usual quantum field theory. As a starting point this special values will get quantum corrections, and if there is no hidden symmetry to protect these special values, they will for sure not survive after we finish the renormalization process. But if we see our models as the bosonic part of theories with supersymmetry, the classical equations of motion for the bosonic fields remain unchanged, and then its solutions, and the form of the potential is automatically given by supersymmetry, and it is protected from quantum corrections via the non renormalization theorems. BPS monopoles and critically coupled vortices arise naturally in supersymmetric theories.

After explaining this concept, we can return to our dualities. We say that a theory exhibits duality if there are two complementary points of view to look at the physical system. There are well and old known dualities in field theory. One for example is the two dimensional sine-Gordon model, that has in its spectrum particles and solitons, and that is completely equivalent to a different theory of interacting fermions known as the Thirring model [41, 42]. The interesting point about this duality is that the particle states of the sine-Gordon theory are mapped to fermion anti-fermion bound states of the Thirring model, and the soliton solutions of the sine-Gordon model are mapped to the fundamental fermion field of the Thirring model. Thus non perturbative process of one theory, like the interaction of the solitons in the sine-Gordon model, and that are very difficult to compute, can be computed using duality as the interaction of two fermions in an interacting field theory, that can be treated perturbatively, and that is by far an easier problem. This property is confirmed by the fact that, if g_s is the coupling of the sine-Gordon system, and g_t is the coupling of the Thirring model, duality relates the two couplings

$$\frac{g_s}{4\pi^2} = \frac{1}{1 + g_t/\pi} \tag{2.116}$$

and we can see that large g_t is mapped to small g_s. Although other known physical systems have dual descriptions, like the two dimensional Ising model, it has been both supersymmetric quantum field theories and string theories who have proposed the most interesting dual descriptions.

For example, one can examine the role of instantons in the $\mathcal{N} = 4$ super Yang-Mills theory under the celebrated AdS/CFT correspondence [43]. The interested reader should consult the original works [44, 45, 46] or a recent review (for example [47]), here we will only mention that integrating the multi instanton measure with the fermion measure that appear in the path integral formalism of the $\mathcal{N} = 4$ super Yang-Mills theory can be interpreted as a integration measure in $AdS_5 \times S^5$. This means, in particular that the integration of this instanton measure against correlators, that is interpreted in the Yang-Mills theory as a instanton computation can be interpreted as a supergravity computation in $AdS_5 \times S^5$. This means that difficult non perturbative contributions in the supersymmetric theory can be computed as a weak coupling gravitational effects. This type of tricks has been used to perform non perturbative computations in other 4D gauge theories [48, 49].

The monopoles are the key idea to understand the electromagnetic duality that has been proposed for certain $\mathcal{N} = 4$ supersymmetric gauge theories (this duality is also sometimes referred as S-duality). This duality in essence says that this theory can be rewritten interchanging the role of the magnetic and the electric field. Under this change, solitonic magnetic monopoles of charge q_m will play the role of electrically charged point particles of charge $q_e = 4\pi/q_m$. The relation between the couplings again is such that maps the strong coupling regime of one side (non perturbative effects) to the small coupling regime of the other (perturbative effects). Again this is an opportunity to get an understanding of the non perturbative effects of strongly coupled non abelian gauge theories, but unfortunately, the electromagnetic duality has not been proved, and is far from being perfectly clear. It seems that a proof of this kind would require a deep knowledge of the non perturbative effects of the supersymmetric gauge theory, that is just what we are looking for, but we can *test* this duality using well known non perturbative effects of the gauge theory: BPS monopoles [50]. For more information about the electromagnetic duality, and the role of BPS monopoles in testing it, the reader can consult the reviews [51, 52], and references therein.

Vortices play a central role in the Seiberg-Intriligator mirror symmetry, where vortices of $\mathcal{N} = 4$, $U(1)$ super Yang-Mills theory with N charged multiplets are mapped into the "electrons" (point particles) of $\mathcal{N} = 4$, $U(1)^{N-1}$ super Yang-Mills with N multiplets. The converse is also true: the electrons of the first theory are mapped to the vortices of the second [53]. Being a duality that maps soliton solutions into particles, this is another example of a strong coupling-weak coupling duality, that can illuminate complex non perturbative problems, like confinement, from a different point of view. In the theory of quantum topological fluids, the duality between vortices and point particles is a well known trick, and the interested reader can consult for example the review [54] and references therein.

Here we have only commented some of the many possible applications that solitons have in both supersymmetric theories and string theory, certainly they play a central role in dualities, and this is another motivation for their study. A very interesting review full of references written from this point of view is D. Tong "TASI lectures on solitons" [55].

2.9 The Bradlow parameter expansion

Vortices will be the main characters of the rest of the work, so here we will only mention their relation with the rest of the topological objects. As we have said, in $3 + 1D$, vortices can be seen as string like objects, and the topological classification of these objects is done by looking at finite energy solutions independent of the coordinate x_3. We will obtain exactly the same results if we study the model in $2+1D$ and look for finite energy configurations, one can always "add" the third dimension and transform the $2 + 1D$ point like objects in $3 + 1D$ string like objects.

As we have argued before, one of the most interesting situations to study solitonic solutions is in the case of supersymmetric theories. The abelian Higgs model can be seen as the bosonic part of a supersymmetric theory in the Bogomolny limit, that has a nice description in terms of interaction between vortices. The Bogomolny limit is achieved by taking the special value of the self coupling of the Higgs field $\lambda = 1$. In superconductors the materials with $\lambda < 1$ are known as *Type I* superconductors, and it has been shown that the interaction between vortices in these superconductors is attractive, thus several vortices in this model tend to join and form a giant vortex. On the other hand, materials with $\lambda > 1$ are known as *Type II* superconductors, and in these materials the force between vortices is repulsive. In these materials vortices form an array maximising their relative distance. The special value $\lambda = 1$, represent the case in which there is no force between vortices, and hence you can put any number of vortices where you want. This statement can be put in a more rigorous way: as we have said in the special case of $\lambda = 1$ the second order equations of motion can be reduced to a set of first order ones, called the Bogomolny equations. These equations will be deduced later, but let us for the moment show them without any proof

$$(D_1 + \imath D_2)\phi = 0 \tag{2.117a}$$

$$B = \frac{1}{2}(1 - |\phi|^2) \tag{2.117b}$$

The non existence of interactions between vortices, and thus the possibility of constructing a solution to these equations by putting the vortices in any place of the plane that we want can be stated as the existence of a unique solution to the Bogomolny equations with flux q given q arbitrary points of the plane (counted with multiplicity). This result was rigorously proved by Taubes [56] for vortices living in $\mathbb{R}^2$, but when we examine this result for the case of vortices living in a compact space, for example the torus, this result is no longer true, as was pointed by Bradlow [57]. The reason is simple: if we integrate the last Bogomolny equation in the Torus, we get

$$2\pi q = \frac{1}{2}\left(\mathcal{A} - \int_{\mathbb{T}^2} d^2x\, |\phi|^2\right) \tag{2.118}$$

where $\mathcal{A}$ is the area of the torus. It is obvious that this last equation cannot have any solution for $\mathcal{A} < 4\pi q$. Bradlow proved that for $\mathcal{A} > 4\pi q$ the space of solutions is also given by the positions of the vortices in the plane. In the special case of $\mathcal{A} = 4\pi q$, there only exists one solution, that can hardly be identified with vortices

$$\phi = 0 \tag{2.119a}$$

$$B = \frac{2\pi q}{\mathcal{A}} \tag{2.119b}$$

These solutions have constant magnetic field, and only exist for the special torus size $\mathcal{A} = 4\pi q$. This resembles the study of self dual configurations in the four dimensional torus, where self dual configurations with constant magnetic field exist only for some special torus sizes. We have called Bradlow parameter to a quantity that measures how far we are from the value $\mathcal{A} = 4\pi q$

$$\epsilon = 1 - \frac{4\pi q}{\mathcal{A}} \tag{2.120}$$

although this is not the same parameter that Bradlow introduced in his original work, it is related to it, and it interpolates between the minimum area that admits solutions to the Bogomolny equations, that is given by $\epsilon = 0$, and the case of vortices living in $\mathbb{R}^2$, that is given by $\epsilon = 1$.

The central idea of the Bradlow parameter expansion is that if the area is close to the critical value $\mathcal{A} = 4\pi q$, the solution to the Bogomolny equations will also be very close to the constant field strength solutions. This is the same strategy used in [32], and explained in the section 2.6.1, to find the self dual configurations in the four dimensional torus.

Although the original work by Bradlow [57] only deals with the case of vortices, but in an arbitrary Kälher manifold, the case of self dual configurations, and vortices share the same common properties: the existence of solutions to the relevant equations but only for some special geometries of the compact space. The idea of perturbing the geometry of the compact space, and find the solutions for arbitrary sizes of the compact space as a perturbative series in a parameter that measures how far we are from the "critical" geometry is the basic idea of the Bradlow parameter expansion, that is the principal result of our work.

The possibilities of this kind of expansion, that we will call generically Bradlow parameter expansions, are far from ending in the results showed in this work. The principal characteristics of this method is that provides a systematic way for computing the solutions, and up to a proof of the convergence of the expansion, it is a demonstration of the existence of these solutions. Usually the first order(s) of the expansion can be computed analytically without too much effort, and then gives an approximate solution to the equations (approximation that can be very good if we are close to the critical geometry). This approximation can be used to explore lots of problems that need an analytical expression for the solutions, and the Bradlow parameter expansion can as well be systematically applied to these other problems. We can use the approximate solutions to compute the Nahm transform of the solutions, to explore the quantum corrections to the classical soliton solutions, to explore the effect that topologically non trivial gauge fields have on the fermions, to compute the metric in the moduli space, etc... There are a huge number of possibilities beyond the results showed in the rest of this work.

One interesting characteristic of these expansions is that although it is usually difficult to compute analytically more than a couple of orders in this expansion, higher order computations can be performed with the help of a computer, and they are not very demanding. This strategy will also be used in the rest of the work, reaching the computation of 51 orders in this expansion.

Having a tool to obtain approximate soliton solutions for arbitrary compact spaces, and for different topological objects, at least instantons and vortices, can be very useful in different areas, and it is the aim of this work to present some possible applications of this method.

2.10 Outline of the thesis

In the following chapter, we will study in detail the Abelian Higgs model, and its classical solutions: the Nielsen-Olesen vortices. There we will see what we know about these solutions: its asymptotic behaviour, an expansion for spherically symmetric solutions, the structure of the moduli space, etc... We will find the Bogomolny equations, a set of first order equations, whose solutions are solutions of the equations of motion for a special value of the Higgs self coupling. As we will see this value of the coupling is the more interesting one, and the Bogomolny equations will play the same role as the self duality equations for instantons. At the end of the chapter, we will also introduce the Extended Abelian Higgs model, in which we have several complex scalar fields, not just one. These models are very interesting, because they arise naturally as the bosonic part of supersymmetric gauge theories [1, 2, 3, 4].

In chapter 4 we will study this model in the two dimensional torus. A key ingredient here is the study of the abelian gauge fields and Higgs fields in the two dimensional torus. Although the main ideas are similar to the ones explored in the section 2.6 the particular case of dealing with an abelian gauge group makes things deserve special attention. The form of the Bogomolny equations and the structure of the moduli space will be the central part of this chapter. The chapter will end with the study of the extended abelian fields on the torus.

Chapter 5 is devoted to finding solutions to the equations of motion. Here the idea of the Bradlow parameter expansion will be developed for the special case of vortices of the Abelian Higgs model in the two dimensional torus. We have computed 51 orders in our expansion, checked the convergence of the series, extrapolated to obtain Nielsen-Olesen vortices and applied the same scheme to find solutions of the extended abelian Higgs model.

In the following chapter, 6, we will study the metric induced by the kinetic energy in the moduli space. This metric gives an interpretation of the collision between vortices as geodesic motion of particles in a manifold with the mentioned metric. This interpretation can be important for cosmic string analysis, and related areas of investigations. As we will see in this chapter, there is not an explicit formula for the metric, only its form for asymptotically separated vortices is well known.

In Chapter 7 we will study this metric for the case of vortices living in the Torus. As we will see the moduli space is very different in the case of vortices living in $\mathbb{R}^2$ and in the torus, so here we will study in detail this case.

Chapter 8 is devoted to obtain the metric for the case of vortices in the torus. Here we will use again the Bradlow parameter expansion, as a method to compute the metric for arbitrary number of vortices with, in principle, the desired precision. As a particular example we will compute 40 orders of the metric for the two vortex case. We will show that near the Bradlow limit, the metric will be completely determined by a handful of numbers that we will give up to machine precision. Later we will extrapolate our data to obtain the metric of vortices in the plane, and as we will see, our computation will agree with the known facts of the metric in this case: the asymptotic behaviour. This data will be used to compute the dynamics, in particular a plot of the scattering angle versus the impact parameter will be shown.

The appendices of this work are an important part of it. Appendix A is a summary of usual conventions that we take along the text. Appendix B is a summary of the most important properties and theorems about the Jacobi theta functions, functions that we use along the text in several places. No prior knowledge of elliptic functions or theta functions is needed to understand this appendix. In appendix C we will see how to construct a special Hilbert space of quasi periodic fields. This appendix is very important, since this quasi periodicity of the fields is the abelian equivalent of the twisted boundary conditions. The particular result of finding a convenient basis for this Hilbert space will be used in several places of the thesis. In appendix D we will develop a technique to obtain the eigenvalues of a matrix that is given as a power series in some parameter. This result will be very important for the study of the eigenvalues of the Dirac operator and the quantum corrections to the classical solutions. In the last appendix, E, the documentation of the `FORTRAN 90` code that we have used to obtain the numerical results of the thesis will be given. In particular the documentation with examples of the code to solve the vortex equation (both for the usual and the extended abelian Higgs model), in the torus for arbitrary position of the zeros of the Higgs field and to compute the metric in the moduli space is given. All the code whose documentation the reader should consult here, is free software, distributed under the terms of the `GPL`[5] license. This means, among other things, that you can use the code any number of times that you want to obtain your own results. The code makes use of the `FORTRAN 90` library `afnl`[6], a numerical library, that can be considered a sub-product of this thesis.

[5] `http://www.gnu.org/copyleft/gpl.html`

[6] The source code and documentation of this library can be found in its sourceforge project page: `http://sourceforge.net/projects/afnl`

Three

The Abelian Higgs model and its classical solutions

3.1 The Abelian Higgs model

The Abelian Higgs model couples an abelian gauge field $A_\mu(x)$ with a scalar charged field ϕ. Once this is said the only point is what is the potential for the scalar field. This potential is chosen so that this model has lot of interesting behaviour like spontaneous symmetry breaking and topological conserved quantities. From a more phenomenological point of view, the Abelian Higgs model can "modelize" the second order phase transition of superconductors, and ϕ can be seen as an order parameter, which describes how deep into the superconducting phase the system is. This is usually called the Ginzburg-Landau theory of superconductivity. Its string-like solutions, that carry magnetic flux through it's core (the vortices) are common objects in physics from condensed matter physics to cosmology and its cosmic strings. From a theoretical point of view, they play a crucial role in some aspects of string theory, and in supersymmetric theories (more details in [55]). Working in $4D$ Minkowski space-time, the Lagrangian of this theory is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu\phi)^*(D^\mu\phi) - \frac{\zeta}{4}\left(|\phi|^2 - \frac{m^2}{\zeta}\right)^2 \tag{3.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength corresponding to the *gauge* field A_μ and $D_\mu = \partial_\mu - ieA_\mu$ is the covariant derivative. This Lagrangian is invariant under the local (gauge) transformation:

$$\phi \longrightarrow e^{ie\alpha(x)}\phi \tag{3.2a}$$

$$A_\mu \longrightarrow A_\mu + \partial_\mu\alpha(x) \tag{3.2b}$$

but the ground state $\phi = e^{i\alpha}\frac{m}{\sqrt{\zeta}}$ is not, so the $U(1)$ local symmetry is spontaneously broken by the Higgs boson $\phi(x)$. Now we want to make a topological analysis of this model, and for reasons that will become clear later, we will work in a $2+1D$ space-time, thus the Latin index $i, j, \cdots = 1, 2$ run only over two space coordinates. We will try to classify the finite energy solutions of the Abelian Higgs model. In order for the energy to be finite, it must happen

$$\lim_{r\to\infty}\left(|\phi|^2 - \frac{m^2}{\zeta}\right) = 0 \Longrightarrow \phi(\infty,\theta) = e^{i\alpha(\theta)}\frac{m}{\sqrt{\zeta}} \tag{3.3a}$$

$$\lim_{r\to\infty} F_{ij}F^{ij} = 0 \Longrightarrow A_i(\infty,\theta) = \partial_i f(\theta) \tag{3.3b}$$

$$\lim_{r\to\infty} D_i\phi = 0 \Longrightarrow \partial_i\alpha(\theta) = e\partial_i f(\theta) \tag{3.3c}$$

where the function $\alpha(x)$ is a multi-valued function (i.e. $\alpha(x) = c, c+2\pi, c+4\pi, \ldots$). This automatically quantises the total magnetic flux that crosses our plane

$$\int B\mathrm{d}^2x = \oint A_i\mathrm{d}x^i = \frac{1}{e}\oint \partial_i\alpha(x)\mathrm{d}x^i = \frac{2\pi q}{e} \tag{3.4}$$

where q is an integer known as the flux number. We have obtained a remarkable result: The configurations with finite energy, must all have quantised flux. This can be understood from a more mathematical point of view: As we had seen, the Higgs field in the infinity is given by:

$$\phi(\infty,\theta) = e^{i\alpha(\theta)}\frac{m}{\sqrt{\zeta}} \tag{3.5}$$

We can see that our Higgs field gives rise to a function that maps points of the spatial infinity (that, since the space is $\mathbb{R}^2$ is the circle S^1) to an element of the gauge group $U(1) \approx S^1$ (see figure 3.1) The Higgs field must have this value at infinity in order for the energy to be finite,

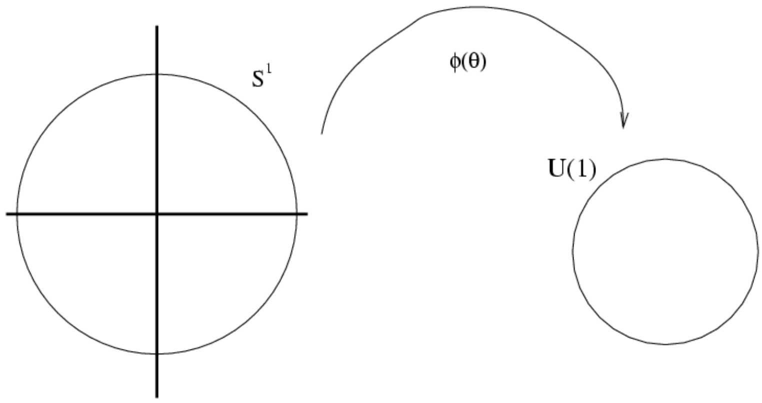


Figure 3.1: ϕ as a map from $\partial\,\mathbb{R}^2 = S^1$ to the gauge group $U(1)$

and the multi-valued function $\alpha(x)$ must be continuous, so really we are looking at the different continuous ways to map a circle into itself. This is a well known mathematical problem (see for example [58, 59, 60]), the fundamental group of the circle ($\pi_1(S^1)$).

Mathematicians know the fundamental groups of almost any interesting space, and for our case we have

$$\pi_1(S^1) = \mathbb{Z} \tag{3.6}$$

This is just the result that we obtained in (3.4) by a more direct approach: The finite energy solutions are classified by an integer. This topological analysis gives us more information, since the fundamental group is a topological property, we know that any continuous deformation of a configuration can not change the flux, no matter if the deformation is a little one or a big one. This also explains why we are interested in string-like configurations: if we look for point like configurations, the topological analysis will not gives us anything interesting, since in that case we will need to compute the different ways in which you can wrap a two sphere in a circle, and in essence there is only one way to do that: the *second* homotopy group of the circle is just the group with one element

$$\pi_2(S^1) = \mathbb{I} \tag{3.7}$$

In the context of our original $4D$ model living in Minkowski space-time, we can say that we have classified the z-independent static solutions (i.e. string like static configurations) with finite energy per unit length by the value of an integer: the magnetic flux that the string carry at its core. Since temporal evolution is a particular case of a continuous deformation, we have that the flux is a conserved quantity. It is a special conserved quantity in the sense that it is not derived from an internal symmetry of the Lagrangian via the Noether theorem. This type of conserved quantity is common in field theory (see for example [39]). This conserved number will also characterise the quantum state of the vortex string in the context of quantum field theory.

3.1.1 Choosing appropiate units

We have the freedom of choosing some units, more appropriate for us. We will use this freedom to simplify the form of our Ginzburg-Landau Lagrangian (3.1) Making the transformation:

$$A_\mu = \chi_1 A'_\mu \tag{3.8a}$$
$$\phi = \chi_2 \phi' \tag{3.8b}$$
$$x^\mu = \frac{1}{\chi_3} x'^\mu \tag{3.8c}$$

the Lagrangian density $\mathcal{L}$, as a function of the "new" variables $(x'^\mu, A'_\mu, \phi', e', m', \zeta')$, transforms into:

$$\mathcal{L} = (\chi_2\chi_3)^2 \left[-\frac{1}{4}\frac{\chi_1^2}{\chi_2^2} F'_{\mu\nu}F'^{\mu\nu} + \frac{1}{2}(D'_\mu\phi')^*(D'^\mu\phi) - \frac{\zeta'}{4}\left(|\phi'|^2 - \frac{m'^2}{\zeta'}\right)^2 \right] \tag{3.9}$$

where the e', m', ζ' are given by

$$e' = \frac{\chi_1}{\chi_3} e \tag{3.10a}$$
$$m' = \frac{1}{\chi_3} m \tag{3.10b}$$
$$\zeta' = \frac{\chi_2^2}{\chi_3^2} \zeta \tag{3.10c}$$

If we take $\chi_1 = \chi_2$, then this rescaling in the fields only changes the Lagrangian by a multiplicative constant. We can choose the scaling constants $\chi_{1,2,3}$ in the following way:

$$\chi_1 = \chi_2 = \frac{m}{\sqrt{\zeta}} \tag{3.11a}$$
$$\chi_3 = \frac{em}{\sqrt{\zeta}} \tag{3.11b}$$

With this choice, the Lagrangian density takes the "standard" form:

$$\mathcal{L} = \frac{m^4 e^2}{\zeta^2}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}|D_\mu\phi|^2 - \frac{\lambda}{8}(|\phi|^2 - 1)^2 \right] \tag{3.12}$$

Now the covariant derivative is given by $D_\mu = \partial_\mu - iA_\mu$, and $\lambda = \frac{2\zeta}{e^2}$. The original form of the Lagrangian density (3.1) is used in Nielsen Olsen [61], our standard form (3.12) is chosen to agree with Rebbi [62]. Every choice of Lagrangian density of the Ginzburg-Landau theory can transformed to our standard form by choosing the scale factors χ_i appropriately.

The action corresponding to the Lagrangian (3.12) is given by

$$S = \int \mathrm{d}^4x \left\{ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu\phi)^*(D^\mu\phi) - \frac{\lambda}{8}(|\phi|^2 - 1)^2 \right\} \tag{3.13}$$

And the equations of motion reads

$$\partial_\mu F^{\mu\nu} = \frac{i}{2}\left[\phi(D^\nu\phi)^* - \phi^* D^\nu\phi\right] \tag{3.14a}$$

$$D^*_\mu D^\mu\phi = -\frac{\lambda}{2}\phi(|\phi|^2 - 1) \tag{3.14b}$$

As we have seen we will be interested in the static $z-$independent solutions of these equations. In that case, we will use the $2D$ equivalent of (3.12)

$$-\mathcal{L} = \mathcal{E} = \frac{1}{4}F_{ij}F^{ij} + \frac{1}{2}|D_i\phi|^2 + \frac{\lambda}{8}(|\phi|^2 - 1)^2 \tag{3.15}$$

where the Latin index $(i, j = 1, 2)$ run only over two space coordinates and $\mathcal{E}$ is the energy density.

3.2 The Nielsen-Olesen Vortex

In this section we will find an approximate asymptotic solution of the equations of motion (3.14) with definite flux, following the article of Nielsen and Olesen [61].

We are looking for spherically symmetric solutions of the equations of motion (3.14) with finite energy (3.15) in $\mathbb{R}^2$. So we will impose that our gauge potential depends only on the radial coordinate r:

$$A_i = \varepsilon_{ij}\partial_j h(r) + \partial_i g(r) \tag{3.16}$$

The magnetic field is given by:

$$F_{ij} = \varepsilon_{ij}B = -\varepsilon_{ij}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[rh'(r)\right] \tag{3.17}$$

Choosing the gauge $\partial_i A_i = 0$ we can eliminate the term $\partial_i g(r)$. Now we are going to use the properties:

$$\partial_i f(r) = \frac{x_i}{r}f'(r) \tag{3.18a}$$

$$\partial_i f(\theta) = -\varepsilon_{ij}\frac{x_j}{r^2}f'(\theta) \tag{3.18b}$$

Although we are looking for spherically symmetric solution, only gauge independent quantities must be spherically symmetric. Therefore we will choose the following ansatz for the Higgs field:

$$\phi = e^{-iq\theta}|\phi| \tag{3.19}$$

where q is the flux number, as we will show in the following section.

3.2.1 Computation of the flux

Since we want finite energy solutions, we must have for $r \longrightarrow \infty$

$$D_i\phi = (\partial_i - iA_i)[e^{-in\theta}|\phi|] = i\varepsilon_{ij}\frac{x_j}{r}\left[\frac{q}{r} - h'(r)\right] = 0 \tag{3.20a}$$

$$F_{ij} = 0 \quad \Longrightarrow \quad h'(r) = \frac{k}{r} \tag{3.20b}$$

where k is some constant. Since both things must be zero at infinity, we must have $k = q$. Now the Flux $\Phi = \int \mathrm{d}^2 x B = \oint \mathrm{d}x^i A_i$ is given by:

$$\Phi = \oint_\infty \mathrm{d}x^i A_i = \int_0^{2\pi} q\left\{r\sin\theta\frac{r\sin\theta}{r^2} + r\cos\theta\frac{r\cos\theta}{r^2}\right\}\mathrm{d}\theta = 2\pi q \tag{3.21}$$

3.2.2 Equations of motion

The equations of motion are:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(rh'(r)\right)\right] = \left[\frac{q}{r} + h'(r)\right]|\phi|^2 \tag{3.22a}$$

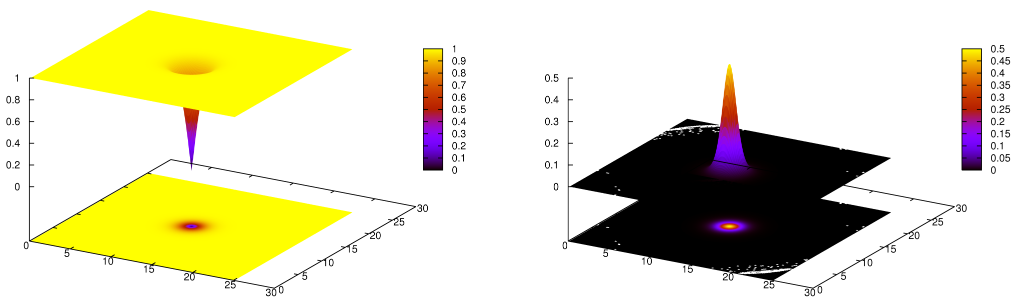
$$\left[h'(r) + \frac{q}{r}\right]\left[h'(r) - \frac{q}{r}\right]|\phi| + \frac{1}{r}|\phi|' + |\phi|'' = \frac{\lambda}{2}|\phi|\left(|\phi|^2 - 1\right) \tag{3.22b}$$

Since $|\phi|$ is gauge invariant, we can write $|\phi| = 1 - \rho(r)$, giving

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(rh'(r)\right)\right] = \left[\frac{q}{r} + h'(r)\right](1 - 2\rho + \rho^2) \tag{3.23a}$$

$$\left[h'(r) + \frac{q}{r}\right]\left[h'(r) - \frac{q}{r}\right](1-\rho) - \frac{1}{r}\rho' - \rho'' = \frac{\lambda}{2}(1-\rho)\left(\rho^2 - 2\rho\right) \tag{3.23b}$$

No explicit solution of this pair of coupled ODE's is known, but we can solve them numerically. In the figures (3.2) we can see the Higgs field and the magnetic field for a $\lambda = 2$ vortex. In the



(a) Higgs field (b) Magnetic field

Figure 3.2: Plots of the Higgs field and the magnetic field for a $\lambda = 2$ vortex.

figure (3.3) we can see the a plot of both the Higgs field and the magnetic field as a function of r.

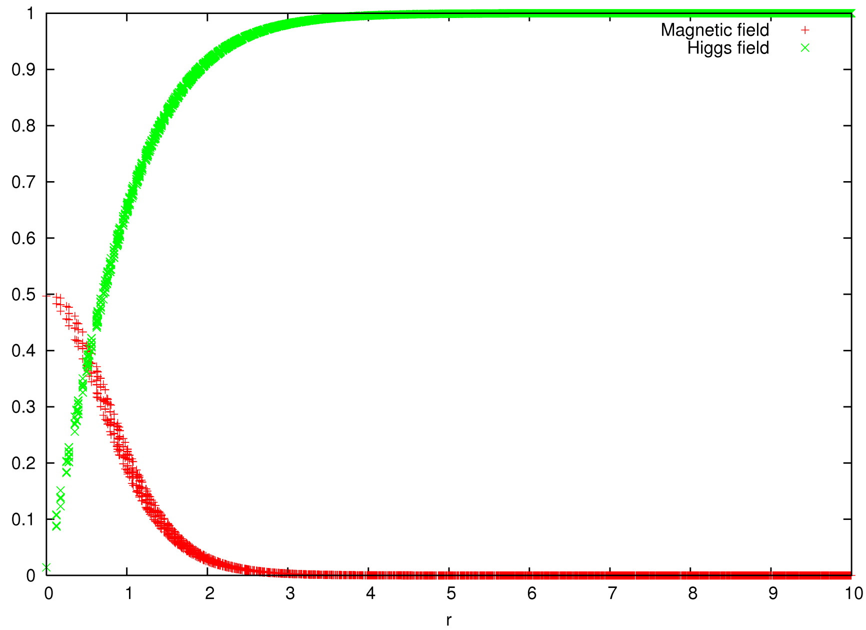


Figure 3.3: Higgs field and magnetic field as a function of r for a $\lambda = 2$ vortex.

3.2.3 Asymptotic behaviour of the solution

We are going to look for the asymptotic behaviour of the solution of the equations of motion. Since we want finite energy configurations, the Higgs field must go to its vacuum expectation value for large r, in order for the energy to be finite, so $\rho \approx 0$ for large r. In this case we can solve the first equation of motion to get:

$$h'(r) = \frac{q}{r} + c\frac{e^{-r}}{\sqrt{r}} + \dots \tag{3.24}$$

This means that the asymptotic behaviour of the magnetic field is given by:

$$B \sim e^{-r} \tag{3.25}$$

The size of the vortex can be roughly estimated by this decrease of the magnetic field, that in our unit is 1, and this is the mass of the photon in the superconductor:

$$m_\gamma = 1 \tag{3.26}$$

Now the mass square of the Higgs particle is given by the second derivative of the potential in the minimum in the Higgs mechanism. This means that:

$$m_H^2 = \lambda \tag{3.27}$$

This can also be compared to the large r approximate solution of the second equation of motion. Neglecting all the $\frac{1}{r}$ terms, and $\rho^n, (n > 1)$, we get

$$\rho'' = \lambda\rho \tag{3.28}$$

So the solutions are $\rho = e^{\pm\sqrt{\lambda}r}$, and we have to choose the $-$ sign to obtain $|\phi| > 0$. Then the asymptotic behaviour of the Higgs field is

$$|\phi| \longrightarrow 1 - e^{-\sqrt{\lambda}r} \tag{3.29}$$

So in our units the mass of the photon is always 1, and the mass of the Higgs field is given by $\sqrt{\lambda}$.

The value $\lambda = 1$ is very special and represent the difference between type I and type II superconductors. In this case the mass of the photon and of the Higgs field are the same, and also in this situation the equations of motion can be reduced to first order ones.

3.3 An expansion for cylindrically symmetric solutions

Here we will review the results of de Vega and Schaposnik [63], where they found an expansion for cylindrically symmetric solutions for a special value of the coupling λ. In principle we have to solve the equations

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(rh'(r)\right)\right] = \left[\frac{q}{r} + h'(r)\right]|\phi|^2 \tag{3.30a}$$

$$\left[h'(r) + \frac{q}{r}\right]\left[h'(r) - \frac{q}{r}\right]|\phi| + \frac{1}{r}|\phi|' + |\phi|'' = \frac{\lambda}{2}|\phi|\left(|\phi|^2 - 1\right) \tag{3.30b}$$

where $A_\theta = rh'(r)$. With the boundary conditions

$$|\phi|(r=0) = 0; h'(0) = 0 \tag{3.31a}$$

$$\lim_{r\to\infty}|\phi| = 1 \tag{3.31b}$$

$$\lim_{r\to\infty}h'(r) = \frac{q}{r} \tag{3.31c}$$

These equations are very difficult to solve, but we will see that they decouple in some limit. In order to do that we will need to compute the Energy-momentum tensor, the non zero components in cylindrical coordinates are given by

$$T_{rr} = \frac{1}{2r^2}\left[\frac{d}{dr}(rh')\right]^2 + \frac{1}{2}\left(\frac{d|\phi|}{dr}\right)^2 - \frac{|\phi|^2}{2r^2}\left(q + rh'\right)^2 - \frac{\lambda}{8}\left(|\phi|^2 - 1\right)^2 \tag{3.32a}$$

$$T_{\theta\theta} = \frac{1}{2}\left[\frac{d}{dr}(rh')\right]^2 - r^2\frac{1}{2}\left(\frac{d|\phi|}{dr}\right)^2 + \frac{|\phi|^2}{2}\left(q + rh'\right)^2 - r^2\frac{\lambda}{8}\left(|\phi|^2 - 1\right)^2 \tag{3.32b}$$

$$T_{00} = \frac{1}{2r^2}\left[\frac{d}{dr}(rh')\right]^2 + \frac{1}{2}\left(\frac{d|\phi|}{dr}\right)^2 + \frac{|\phi|^2}{2r^2}\left(q + rh'\right)^2 + \frac{\lambda}{8}\left(|\phi|^2 - 1\right)^2 \tag{3.32c}$$

and the conservation laws are given by

$$\frac{d}{dr}\left(rT_{rr}\right) = \frac{T_{\theta\theta}}{r^2} \tag{3.33}$$

now we can make the ansatz (compatible with the conservation laws)

$$T_{rr} = T_{\theta\theta} = 0 \tag{3.34}$$

Using this ansatz, we obtain the equations

$$\frac{1}{r^2}\left[\frac{d}{dr}(rh')\right]^2 = \frac{\lambda}{8}\left(|\phi|^2 - 1\right)^2 \tag{3.35a}$$

$$\left(\frac{d|\phi|}{dr}\right)^2 = \frac{|\phi|^2}{r^2}\left(q + rh'\right)^2 \tag{3.35b}$$

the second of these equations is separable, and can be solved, obtaining

$$|\phi| = \exp\left\{\pm\int_0^r \frac{dx}{x}(q + xh')\right\} \tag{3.36}$$

and we have to choose the plus sign to obtain the correct boundary conditions (3.31). With this condition the equations of motion are reduced to

$$\frac{1}{r}\frac{d}{dr}(rh') = \pm\sqrt{\frac{\lambda}{8}}\left(|\phi|^2 - 1\right) \tag{3.37a}$$

$$\frac{d|\phi|}{dr} = \frac{|\phi|}{r}\left(q + rh'\right) \tag{3.37b}$$

It is easy to show that any solution of (3.37), is also a solution of the equations of motion (3.30) iif $\lambda = 1$. In fact in this case the equations decouple, and equations (3.37) are equivalent to

$$\frac{d^2}{dr^2}(rh') - \frac{1}{r}\frac{d}{dr}(rh') - (q + rh')\left[\frac{\sqrt{2}}{2} + \frac{2}{r}\frac{d}{dr}(rh')\right] = 0 \tag{3.38a}$$

$$|\phi|^2 = 1 + \frac{2\sqrt{2}}{r}\frac{d}{dr}(rh') \tag{3.38b}$$

we will rescale the coordinate r, defining $\rho = \frac{r}{2\sqrt[4]{2}}$, and we will also define $H = q + rh'$. Then equations (3.39) become

$$\frac{d^2H}{d\rho^2} - \frac{1}{\rho}\frac{dH}{d\rho} - 4H = 2\frac{H}{\rho}\frac{dH}{d\rho} \tag{3.39a}$$

$$|\phi|^2 = 1 + \frac{1}{2\rho}\frac{dH}{d\rho} \tag{3.39b}$$

The boundary conditions for $H(\rho)$ are

$$H(0) = q; \qquad \lim_{\rho\to\infty} H(\rho) = 0 \tag{3.40}$$

since the first of these equations is invariant under the transformation $\rho \to -\rho$, we can make a power expansion of $H(\rho)$ in even powers of ρ

$$H(\rho) = q + \sum_{s=1}^{\infty} D_q^s \rho^{2s} \tag{3.41}$$

$\mathbf{D_1^1}$	$\mathbf{D_1^2}$	$\mathbf{D_1^3}$	$\mathbf{D_1^4}$	$\mathbf{D_1^5}$
-1.0	7.27910×10^{-1}	-4.85273×10^{-1}	3.14441×10^{-1}	-2.01596×10^{-1}
$\mathbf{D_1^6}$	$\mathbf{D_1^7}$	$\mathbf{D_1^8}$	$\mathbf{D_1^9}$	$\mathbf{D_1^{10}}$
1.28657×10^{-1}	-8.19221×10^{-2}	5.20996×10^{-2}	-3.31095×10^{-2}	2.10315×10^{-2}

Table 3.1: deVega and Schaposnik coefficients for the one-vortex solution

the subscript q in D_q^s means that these coefficients are different for different values of the flux number. The recursion relation for these coefficients are

$$D_q^1 = -1 \tag{3.42a}$$

$$D_q^s = 0 \quad (2 \leq s \leq q) \tag{3.42b}$$

$$D_q^s = \frac{1}{s(s-q-1)} \sum_{l=q+1}^{s-1} l D_q^l D_q^{s-l} \quad (s \geq q+2) \tag{3.42c}$$

the first of these equations implements the boundary condition at $\rho = 0$, and the recursion relation allows us to obtain all the coefficients in terms of D_q^1 and D_q^{q+1}. We do not know the value of D_q^{q+1}, and it has to be related with the boundary condition at infinity. To obtain it we will use an integral equation equivalent to equation (3.39a). Taking into account that the left hand side of equation (3.39a) can be related with the Bessel equation of order one, we obtain

$$\begin{aligned} H(\rho) &= 2q\rho K_1(2\rho) \\ &- 2\rho K_1(2\rho) \int_0^\rho dx I_1(2x) \frac{H(x)}{x} \frac{dH}{dx} \\ &- 2\rho I_1(2\rho) \int_0^\rho dx K_1(2x) \frac{H(x)}{x} \frac{dH}{dx} \end{aligned} \tag{3.43}$$

The small ρ solutions of this integral equation and of equation (3.39a) are the same, and this fact can be used to obtain numerically the value of D_q^{q+1}. Here we will skip the details (the interested reader can should read [63]), and switch to the most practical problem of obtaining the coefficients for a representative example

3.3.1 The explicit one-vortex solution

For the one vortex case, the coefficient D_1^1 and D_1^2 is given by

$$\begin{aligned} D_1^1 &= -1 \\ D_q^2 &= 0.72791... \end{aligned} \tag{3.44}$$

and these are the coefficients that we need to start the recursion. In the table (3.1) we can see the first ten coefficients. Once this coefficients are known, we can reconstruct both the Higgs field and the Magnetic field, with the help of the formulas

$$B(r) = -2 \sum_s s D_1^s \left(\frac{1}{4\sqrt{2}}\right)^s r^{2(s-1)} \tag{3.45}$$

$$|\phi(r)|^2 = 1 + 4\sqrt{2} \sum_s s D_1^s \left(\frac{1}{4\sqrt{2}}\right)^s r^{2(s-1)} \tag{3.46}$$

3.4 The Bogomolny Equations

In the context of superconductors, the value $\lambda = 1$ is a very special one, but there are superconductors in nature with a value of λ very close to this special value. Superconductors with $\lambda < 1$ are also called *Type I superconductors*, and superconductors with $\lambda > 1$ are called *Type II superconductors*. The interaction between flux vortices in type I superconductors is attractive [64], thus several vortices in a type I superconductor tend to join together in a giant vortex that carries all the flux (see figure (3.5)). On the other hand in type II superconductors

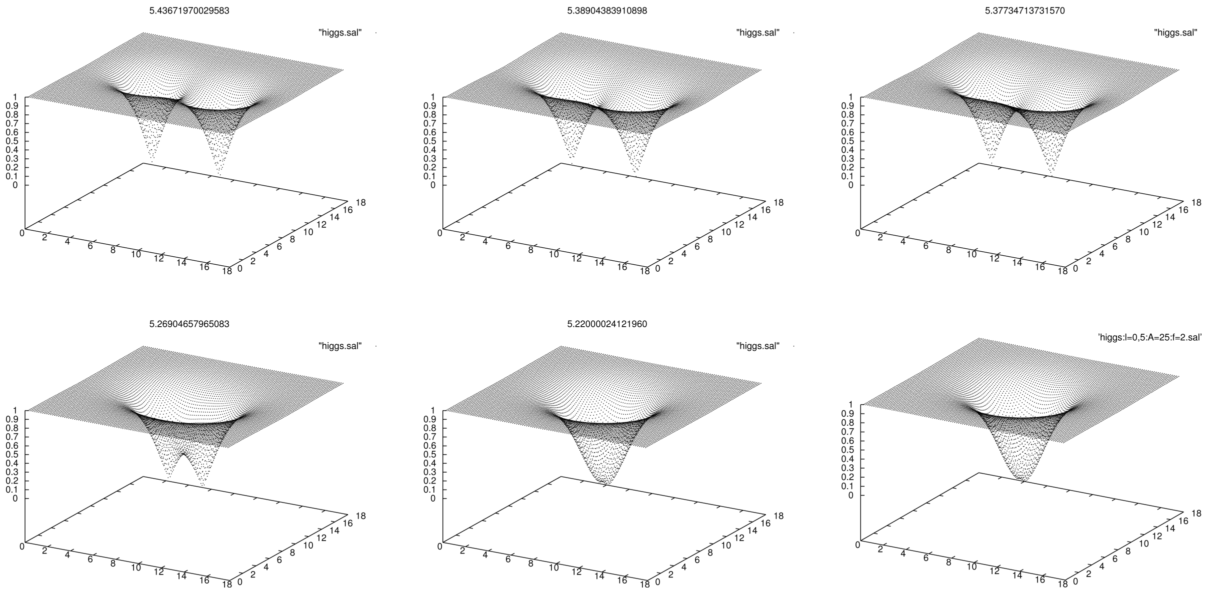


Figure 3.4: Attractive interaction of the vortex for a Type I superconductor ($\lambda = 0.5$ in this example). We can see how the energy of the configurations decreases with the distance between vortices.

the interaction between flux vortices is repulsive, then, several vortices in a finite superconductor tend to organise themselves in arrays of vortices that maximise the distance between them [65, 64] (see figure (3.4)). In the Bogomolny limit (in which $\lambda = 1$) the interaction between static vortices is null, and then we can place the vortices where we want (we will study this case in detail later). Although Bogomolny vortices in the static case are non interacting objects, they still have a very interesting dynamics when they move. This interaction of moving vortices can describe the collisions of vortices strings in the superconductors, or the collisions of cosmic strings. This subject will be the studied in the chapters (6, 7 and 8).

In the context of field theories and cosmic strings, the value $\lambda = 1$ is very important, because this is just that the value that the supersymmetric abelian Higgs model needs to maintain the supersymmetries. This makes the study of Bogomolny equations, and of the interactions of Bogomolny vortices very important.

We have just seen, that the $\lambda = 1$ case is very special: the equations of motion for solutions with cylindrical symmetry decouple. Here we will see that in general the second order equations of motion, can be reduced to a set of first order ones, known as the Bogomolny equations. The reduction is obtained by finding a lower bound to the energy, that reads

$$E = \frac{1}{2}\int \mathrm{d}^2x \left\{ B^2 + |D_i\phi|^2 + \frac{1}{4}(|\phi|^2 - 1)^2 \right\} \tag{3.47}$$

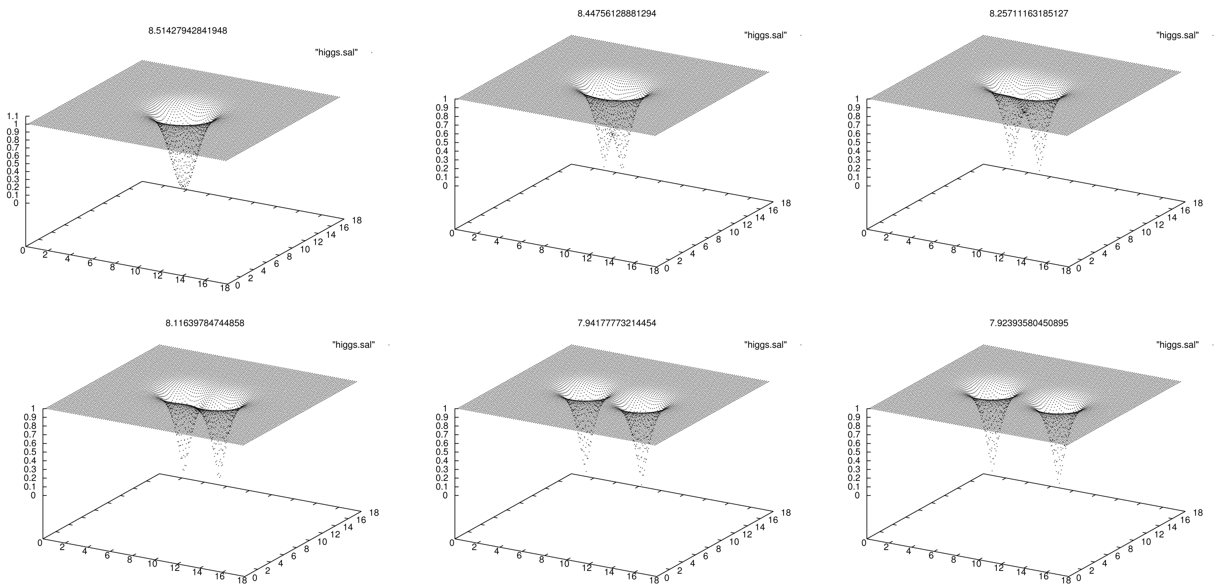


Figure 3.5: Repulsive interaction of the vortex for a Type II superconductors ($\lambda = 3$ for this example). We can see how the energy of the configurations decreases with the distance between vortices.

To find this bound we will proceed as follow: we consider the quantity D_ϵ defined by:

$$D_\epsilon = D_1 + \imath\epsilon D_2 \tag{3.48}$$

where $\epsilon = \pm 1$, we can compute

$$|D_\epsilon \phi|^2 = |(D_1 + \imath\epsilon D_2)\phi|^2 = |D_1\phi|^2 + |D_2\phi|^2 - \imath\epsilon \left[D_2^*\phi^* D_1\phi - D_1^*\phi^* D_2\phi\right] \tag{3.49}$$

We can integrate by parts the quantity in brackets to obtain:

$$\begin{aligned}
[\dots] &= (\partial_2 + \imath A_2)\phi^*(\partial_1 - \imath A_1)\phi - (\partial_1 + \imath A_1)\phi^*(\partial_2 - \imath A_2)\phi \\
&= \imath \left[A_2\phi^*\partial_1\phi - A_1\phi\partial_2\phi^* - A_1\phi^*\partial_2\phi + A_2\phi\partial_1\phi^*\right] + \partial_2\phi^*\partial_1\phi - \partial_1\phi^*\partial_2\phi \\
&= \imath \left[-\partial_1 A_2|\phi|^2 - A_2\phi\partial_1\phi^* + \partial_2 A_1|\phi|^2 + A_1\phi^*\partial_2\phi - A_1\phi^*\partial_2\phi + A_2\phi\partial_1\phi^*\right] + \text{surface terms} \\
&= -\imath(\partial_1 A_2 - \partial_2 A_1)|\phi|^2 + \text{surface terms} \\
&= -\imath B|\phi|^2 + \text{surface terms}
\end{aligned} \tag{3.50}$$

So we can write:

$$\frac{1}{2}|D_\epsilon\phi|^2 = \frac{1}{2}|D_1\phi|^2 + \frac{1}{2}|D_2\phi|^2 - \epsilon\frac{B}{2}|\phi|^2 + \text{surface terms} \tag{3.51}$$

Inserting this into the equation for the energy, we obtain (the surface terms vanish under the integration)

$$\begin{aligned}
E &= \frac{1}{2}\int \mathrm{d}^2x \left\{ B^2 + \epsilon B|\phi|^2 + \frac{1}{4}(|\phi|^2 - 1)^2 + |D_\epsilon\phi|^2 \right\} \\
&= \frac{1}{2}\int \mathrm{d}^2x \left\{ \left[B + \epsilon\frac{1}{2}(|\phi|^2 - 1)\right]^2 + \epsilon B + |D_\epsilon\phi|^2 \right\}
\end{aligned} \tag{3.52}$$

Since both $|D_\epsilon|$ and $\left[B + \epsilon\frac{1}{2}(|\phi|^2 - 1)\right]^2$ are greater than zero we can derive a lower bound for the energy in the following way. If the flux is greater than zero we choose $\epsilon = 1$, and then

$$E \geq \frac{1}{2}\int \mathrm{d}^2 x B = q\pi \tag{3.53}$$

This bound is saturated iif

$$(D_1 + \imath D_2)\phi = 0 \tag{3.54a}$$

$$B = \frac{1}{2}(1 - |\phi|^2) \tag{3.54b}$$

If the flux is negative, we choose $\epsilon = -1$, and then

$$E \geq -\frac{1}{2}\int \mathrm{d}^2 x B = |q\pi| \tag{3.55}$$

This bound is saturated iif

$$(D_1 - \imath D_2)\phi = 0 \tag{3.56a}$$

$$-B = \frac{1}{2}(1 - |\phi|^2) \tag{3.56b}$$

In general we can write this equations as:

$$E \geq |q\pi| \tag{3.57a}$$

$$(D_i + \imath\epsilon D_2)\phi = 0 \tag{3.57b}$$

$$|B| = \frac{1}{2}(1 - |\phi|^2) \tag{3.57c}$$

where $\epsilon = \mathrm{sign}(\Phi)$. These are the Bogomolny equations. We can always choose the flux positive (it is a matter of conventions), and then we usually will use the positive flux form of these equations.

For values of $\lambda \neq 1$, we can also write a bound for the energy in the following way

$$E \geq |q\pi| + \frac{\lambda - 1}{8}\int \mathrm{d}^2 x(|\phi|^2 - 1)^2 \tag{3.58}$$

note that now the lower bound of $|q\pi|$ can never be reached but for the special case $q = 0$ using the "vacuum" solution ($\phi = 1$ and $B = 0$). With the help of this bound, we expect that minimum energy configurations will have $E > q\pi$ for $\lambda > 1$, and $E < q\pi$ for $\lambda < 1$.

3.5 Solutions of the Bogomolny equations

The characterisation of the solutions of the Bogomolny equations was first studied by Taubes [56, 66, 67], and he proved some interesting facts. First he showed that using the first Bogomolny equation, the gauge potential can be eliminated from the Bogomolny equations. Writing the Higgs and gauge fields as

$$\phi = e^H \tag{3.59a}$$

$$A_1 = \frac{1}{2}\partial_2 h + \partial_1 g \tag{3.59b}$$

$$A_2 = -\frac{1}{2}\partial_1 h + \partial_2 g \tag{3.59c}$$

where $H = \frac{1}{2}h + \imath g$ is easy to see that the first Bogomolny equation is solved, and the second one is given by

$$\nabla^2 h + 1 - e^h = 0 \tag{3.60}$$

This equation is valid everywhere but at the zeros of the Higgs field, where h becomes singular. It is easy to check that near a simple zero of the Higgs field $h \approx 2\log|x - x_r|$, and using

$$\nabla^2 \log|x - x_r|^2 = 4\pi\delta^2(x - x_r) \tag{3.61}$$

The second Bogomolny equation including sources (known as the *vortex equation*) is given by

$$\nabla^2 h + 1 - e^h = 4\pi \sum_{r=1}^{q} \delta^2(x - x_r) \tag{3.62}$$

Thus all the properties of the solutions of the Bogomolny equations should be obtained from this vortex equation.

Second he proved that the set of zeros of the Higgs field is discrete. The interested reader should consult [56, Appendix].

Third he proved that given q not necessarily different points x_r $(r = 1, \ldots, q)$ in $\mathbb{R}^2$, there exist a unique solution with flux number q to the vortex equation such that the Higgs field has precisely q zeros (counted with multiplicity) located at x_r. To prove that we will define a function (note that we use the complex notation of appendix A.4)

$$u_1 = -\sum_{r=1}^{q} \ln\left(1 + \frac{\mu}{|z - z_r|^2}\right) \tag{3.63}$$

with $\mu \in \mathbb{R}$ a real number. On $\mathbb{C}\backslash\{z_r\}$, u_1 is C^∞. Now we define two functions: first

$$g_1 = 4\sum_{r=1}^{\infty} \frac{\mu}{(|z - z_r|^2 + \mu)^2} \tag{3.64}$$

and second

$$v = 2h + u_1 \tag{3.65}$$

Now the vortex equation can be rewritten as

$$\Delta v + g_1 - 1 - e^{u_1}e^v = 0 \tag{3.66}$$

with the boundary condition

$$\lim_{|x|\to\infty} v = 0 \tag{3.67}$$

Equation (3.66) can be seen as the Euler-Lagrange equations of the variational problem associated with the functional

$$\mathcal{A}[v] = \int_{\mathbb{R}^2} d^2x \left\{\frac{1}{2}\partial_\mu v \partial_\mu v + v(1 - g_1) - e^{u_1}(e^v - 1)\right\} \tag{3.68}$$

The path that leads to the desired proof has two steps

1. $\mathcal{A}[v]$ is strictly convex.

2. For a large enough ball in function space, the normal derivative of $\mathcal{A}[v]$ on the boundary of the ball is positive.

that using standard results of functional analysis (see [68])imply the existence and uniqueness of the minimum of $\mathcal{A}[v]$. The interested reader should consult the details of the proof in [56].

This result of Taubes, allows us to interpret the solutions of the vortex equations as a superposition of q vortices located at arbitrary points in the plane (the zeros of the Higgs field).

Despite of the fact that the general properties of the solutions of the Bogomolny equations are well known, there does not exist explicit solutions (but the cylindrically symmetric solutions of section (3.3)), as in the case of the instantons or the monopoles. In general, solutions have to be constructed numerically. First you have to choose q points in the plane $\{x_r\}$, put by hand $h = 2\log\epsilon$ in a small circle of radius ϵ around each zero (if the multiplicity of the zero is bigger $n > 1$, then you have to write $h = 2n\log\epsilon$), and $h = 0$ in a circle far from any zero of the Higgs field (this implement the condition that the Higgs field should go to its vacuum expectation value at infinity). Then the vortex equation has to be solved in the region between the zeros and this circle.

3.5.1 The moduli space

As we have said, given any q points in the plane $\{x_r\}$ there exist a unique solution to the Bogomolny equations, so the moduli space of solutions are the unordered q-tuples of points in the plane $(\mathbb{R}^2)^q/S_q$ (S_q is the permutation group of q elements). We have another way to characterise the moduli. Given q points of the complex plane $\{x_r\}$, we can form the q complex numbers $Z_r = x_{1,r} + \imath x_{2,r}$, and there is a one to one correspondence between the complex numbers Z_r, and the coefficients $\{a_i\}$ of the polynomial

$$p(z) = \prod_{i=1}^{q} (z - Z_i) = a_0 + a_1 z + \cdots + a_q z^{q-1} + z^q \tag{3.69}$$

so the q *ordered* complex numbers $\{a_i\}$ also gives us a parametrisation of the moduli space, showing that can be identified with $\mathbb{C}^q$.

3.6 Extended Abelian Higgs Models

Here we will study some models related with the normal Abelian Higgs Model. In these models, the scalar field ϕ, is replaced with a vector of $M = N + N'$ components[1]

$$\phi_A(x) \qquad (A = 1, \ldots, M) \tag{3.70}$$

[1] In this section the indices of the beginning of the Latin alphabet will run over the numbers $1, \ldots, N$, indices of the end of the Latin alphabet will run over the numbers $N+1, \ldots, N+N'$, and the upper case indices of the beginning of the Latin alphabet will run over the complete set $1, \ldots, M$. i.e.

$$\begin{aligned} a, b, c, \ldots &= 1, \ldots, N \\ r, s, t, \ldots &= N+1, \ldots, M = N + N' \\ A, B, C, \ldots &= 1, \ldots, M = N + N' \end{aligned}$$

the Lagrangian will be constructed so that the model has the usual local $U(1)$ gauge symmetry, but also a global $SU(N) \times SU(N')$ symmetry

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu\phi_A)^*(D^\mu\phi_A) - \frac{\lambda}{8}\left(|\phi_a|^2 - |\phi_r|^2 - 1\right)^2 \tag{3.71}$$

where the sum over the indices a, r and A is implied. The symmetry group of this Lagrangian is $SU(N)_{\text{global}} \times SU(N')_{\text{global}} \times U(1)_{\text{local}}$. Under a global transformation given by $U_{AB} \in SU(N) \times SU(N')$, the fields transforms in the following way

$$\phi_A \to U_{AB}\phi_B; \qquad A_\mu \to A_\mu \tag{3.72a}$$

and under the local $U(1)$ symmetry given by $e^{\imath\alpha(x)}$

$$\phi_A \to e^{\imath\alpha(x)}\phi_A; \qquad A_\mu \to A_\mu + \partial_\mu\alpha(x) \tag{3.72b}$$

This model has also string like solutions [3] of the equations of motion, whose core carry a magnetic flux that is quantised, as we will see. These string-like solutions, usually called *semilocal strings*, are stable [2, 1], and as we will see, the model is richer than the usual Abelian Higgs Model. These models are very interesting from the theoretical point of view, because they arise naturally as the bosonic part of supersymmetric models.

We are ready to derive the bound for the energy, and the Bogomolny equations. We will only derive the positive flux form of the Bogomolny equations (the derivation of the negative flux form of the equations is straightforward). The energy of this model (in $2+1D$) for the special value $\lambda = 1$ reads

$$E = \frac{1}{2}\int \mathrm{d}^2x \left\{ B^2 + (D_i\phi_A)^*(D_i\phi_A) + \frac{1}{4}(|\phi_a|^2 - |\phi_r|^2 - 1)^2 \right\} \tag{3.73}$$

as in the previous case, we will define

$$D_\pm = D_1 \mp \imath D_2 \tag{3.74}$$

the key to obtain a useful lower bound is that we need to complete the square. For that purpose, we need to write

$$E = \frac{1}{2}\int \mathrm{d}^2x \left\{ \left[B + \frac{1}{2}(|\phi_a|^2 - |\phi_r|^2 - 1)\right]^2 + |D_-\phi_a|^2 + |D_+\phi_r|^2 + B \right\} \tag{3.75}$$

Now since all the terms inside the integral sign are positive, we have a lower bound to the energy

$$E \geq \frac{1}{2}\int \mathrm{d}^2x B = q\pi \tag{3.76}$$

the bound is saturated iif

$$D_-\phi_a = (D_1 + \imath D_2)\phi_a = 0 \qquad (a = 1, \ldots, N) \tag{3.77a}$$

$$D_+\phi_r = (D_1 - \imath D_2)\phi_r = 0 \qquad (r = N+1, \ldots, M) \tag{3.77b}$$

$$B = \frac{1}{2}\left(1 - |\phi_a|^2 + |\phi_r|^2\right) \tag{3.77c}$$

and these are the Bogomolny equations of the Extended Abelian Higgs Model. The first thing that we see, is that using a solution of the (non extended) abelian Higgs model we can construct solutions of this extended model. For that purpose, and calling $(\phi(x), A_i(x))$ the higgs and gauge field of such a solution, we write

$$\begin{aligned}\phi_a &= \mathcal{N}_a\phi \\ \phi_r &= 0 \\ B &= \partial_1 A_2 - \partial_2 A_1\end{aligned} \tag{3.78}$$

where $\mathcal{N}_a$ are complex constants, such that

$$\sum_a |\mathcal{N}_a|^2 = 1 \tag{3.79}$$

In general we will have other solutions of the Extended Abelian Higgs model, that cannot be constructed in this way. In this sense the Extended Abelian Higgs Model is richer than the non extended one.

3.6.1 Reduction of the Bogomolny equations

As we have seen, we can write a general gauge potential in the following way

$$\begin{aligned}A_1 &= \frac{1}{2}\partial_2 h + \partial_1 g \\ A_2 &= -\frac{1}{2}\partial_1 h + \partial_2 g\end{aligned} \tag{3.80a}$$

using complex notation (for the details about this notation see appendix A), we can write this gauge potential as

$$A = \imath\partial\overline{H} \tag{3.81}$$

where $H = \frac{1}{2}h + \imath g$. This means that the first type of Bogomolny equations

$$(D_1 + \imath D_2)\phi_a = 0 \tag{3.82}$$

is translated into

$$(\overline{\partial} + \overline{\partial}H)\phi_r = 0 \tag{3.83}$$

and the second type

$$(D_1 - \imath D_2)\phi_r = 0 \tag{3.84}$$

into

$$(\partial + \partial\overline{H})\phi_a = 0 \tag{3.85}$$

If we write

$$\begin{aligned}\phi_a &= \mathcal{N}_a e^{-\overline{H}}\eta_a \\ \phi_r &= \mathcal{N}_r e^{H}\eta_r\end{aligned} \tag{3.86}$$

where $\mathcal{N}_A$ are complex constants, equations (3.83) and (3.85) are reduced to

$$\begin{aligned}\overline{\partial}\eta_a = 0 \quad &\Longrightarrow \quad \eta_a = \eta_a(z) \\ \partial\eta_r = 0 \quad &\Longrightarrow \quad \eta_r = \eta_r(\overline{z})\end{aligned} \tag{3.87}$$

so the functions $\eta_a(z)$ are analytic functions of z, and the functions $\eta_r(\overline{z})$ are analytic functions of $\overline{z}$. By Liouville theorem there does not exist any analytic bounded function in the complex plane. This means that

$$\lim_{|z|\to\infty} \eta_A = \infty \tag{3.88}$$

To have finite energy solutions $e^{\pm\frac{1}{2}h}$ must go to zero at infinity, so that the Higgs field has a finite value when $|z| \to \infty$. But it is impossible that, at the same time $\lim_{|z|\to\infty} e^{\frac{1}{2}h} = 0$ and $\lim_{|z|\to\infty} e^{-\frac{1}{2}h} = 0$. This means that for the energy to be finite, one of the two types of constants $\mathcal{N}_a$ or $\mathcal{N}_r$ must be zero. To know which kind of constants are zero, we make use of the other Bogomolny equation. Imagine first that $\mathcal{N}_a = 0$. Then the last Bogomolny equation reads

$$B = \frac{1}{2}\left(1 + |\phi_r|^2\right) \tag{3.89}$$

integrating both sides of this equation

$$4\pi q = \int_{\mathbb{R}^2} d^2x \, \left(1 + |\phi_r|^2\right) \tag{3.90}$$

But this is clearly impossible, since the integrand is always greater than one. So it must be $\mathcal{N}_r = 0$. This means that the solutions of the Bogomolny equations for the extended abelian Higgs model with a symmetry group $SU(N)_{\mathrm{global}} \times SU(N')_{\mathrm{global}} \times U(1)_{\mathrm{local}}$ are exactly the same as the solutions of the extended abelian Higgs model with a symmetry group $SU(N)_{\mathrm{global}} \times U(1)_{\mathrm{local}}$. In other words, the only solution of the equation

$$(D_1 - \imath D_2)\phi_r = 0 \tag{3.91}$$

compatible with the other Bogomolny equations is

$$\phi_r = 0 \tag{3.92}$$

An example of this can be found in the usual Nielsen Olesen vortex. In the section 3.2 we have proved that for cylindrically symmetric solutions the asymptotic behaviour of the gauge connection is given by

$$h'(r) \approx \frac{q}{r} \Longrightarrow h(r) \approx q\log(r) \tag{3.93}$$

and, as we expect

$$\lim_{r\to\infty} e^{-\frac{1}{2}h} = 0 \tag{3.94}$$

An open and interesting question is whether this behaviour is also true for the general equations of motion, that is to say, not for the special case $\lambda = 1$, when all the information is in the Bogomolny equations, but for the general case of arbitrary λ. Also the same question is open when the different Higgs fields have different charges, in this case we do not have a global $SU(N) \times SU(N')$ symmetry, but the question is still interesting, since in this case it is impossible to write a bound for the energy of the Bogomolny type. Only numerical work, and for the special case of cylindrically symmetric solutions, has been done in this direction [69], but a general analytic proof is still an open question.

3.6.2 Topological analysis

Having reduced the Bogomolny equations, we will repeat here the topological argument of the usual abelian Higgs model. We will work with the reduced model

$$E = \frac{1}{2}\int \mathrm{d}^2x \left\{ B^2 + (D_i\phi_a)^* (D_i\phi_a) + \frac{1}{4}(|\phi_a|^2 - 1)^2 \right\} \tag{3.95}$$

in order for the energy to be finite, the Higgs field must lie in its vacuum manifold at infinity. This manifold is given by

$$\mathcal{V} = \left\{ |\phi_a|^2 = 1 \right\} = S^{2N-1} \tag{3.96}$$

and is simply connected. The reason why we still have a non trivial topology associated with the finite energy solutions is that the finiteness of the kinetic term

$$\lim_{|x_i|\to\infty} D_i\phi_a = 0 \tag{3.97}$$

requires that the Higgs field varies at infinity at most by a phase, effectively reducing the manifold where the Higgs field can lie at infinity to a circle, that is not simply connected

$$\pi_1(S^1) = \mathbb{Z} \tag{3.98}$$

Once more, finite energy solutions are classified by an integer (the flux number). A particular case of this is when $N = 2$, then the vacuum manifold S^3 fibres as a S^1 bundle over S^2 (the well known Hopf bundle of S^3).

3.6.3 The moduli space of solutions with $N = 2$

The special case $N = 2$ is the one that has deserved more attention in the literature, and is this case the one that we will study in detail here. As we have said the Bogomolny equations for this model are

$$(D_1 + \imath D_2)\phi_a = 0 \tag{3.99a}$$
$$B = \frac{1}{2}\left(1 - |\phi_a|^2\right) \tag{3.99b}$$

in the points where $\phi_a \neq 0$, and using complex notation, we can write

$$A = \imath\partial \ln \overline{\phi}_a \tag{3.100}$$

Subtracting these two equations, we obtain

$$\overline{\partial} \ln w(z) = 0 \tag{3.101}$$

where

$$w(z) = \frac{\phi_2}{\phi_1} \tag{3.102}$$

is locally analytic in z. We can eliminate the gauge potential A_i from the remaining Bogomolny equation

$$\Delta f + 1 - e^f(1 + |w|^2) = 0 \tag{3.103}$$

where $f = \ln|\phi_1|^2$. This equation is valid everywhere except at the zeros of ϕ_1. Since ϕ_1 has the zeros at discrete points in the plane, near such a zero Z_0 of multiplicity n_0, the Higgs field is given by

$$\phi_1(x) = (z - Z_0)^{n_0} g(x) \tag{3.104}$$

where $g(x)$ is a smooth non vanishing function. Since the global $SU(2)$ symmetry allows us to write, without loss of generality, the particular asymptotic form

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \longrightarrow \begin{pmatrix} e^{\imath\alpha(\theta)} \\ 0 \end{pmatrix} \tag{3.105}$$

we have that $w(z)$ must vanish at infinity. Using the methods of Taubes [67] we can prove that ϕ_1 has precisely q zeros (counted with multiplicity). Denoting this zeros Z_r ($r = 1, \dots, q$) it follows that

$$w(z) = \frac{Q(z)}{P(z)} \tag{3.106}$$

where $Q(z)$ and $P(z)$ are polynomials of degree q and $q-1$ respectively

$$P(z) = \prod_{r=1}^{q} (z - Z_r) = z^q + \sum_{k=0}^{q-1} a_k z^k \tag{3.107a}$$

$$Q(z) = \sum_{k=0}^{q-1} q_k z^k \tag{3.107b}$$

Since in two dimensions

$$\Delta \ln|z - Z_r|^2 = 4\pi\delta(x - x_r) \tag{3.108}$$

the generalisation of equation (3.103) to be valid at the zeros of ϕ_1 is

$$\Delta f + 1 - e^f(1 + |w|^2) = 4\pi \sum_{i=r}^{q} \delta(x - x_r) \tag{3.109}$$

with boundary conditions

$$\lim_{|x|\to\infty} f(x) = 0 \tag{3.110}$$

Writing $u = f + \ln(1 + |w|^2)$ equation (3.103) becomes

$$\Delta u + 1 - e^u = \rho \tag{3.111}$$

with boundary conditions

$$\lim_{|x|\to\infty} u(x) = 0 \tag{3.112}$$

and ρ given by

$$\rho = \Delta \ln(|P|^2 + |Q|^2) \tag{3.113}$$

The problem is reduced to that of studying the solutions of equation (3.111), and a minor modification of the work of Taubes [67] described in section 3.5 also works here [70], the only difference is that now the function u_1 is defined as

$$u_1 = \ln(|P|^2 + |Q|^2) - \sum_{r=1}^{q} \ln(|z - Z_r|^2 + \mu) \tag{3.114}$$

and the rest of the proof works in a similar way, thus proving the existence and uniqueness of a solution to this modified vortex equation for any given polynomials $P(z)$ and $Q(z)$. This means that the moduli space is characterised by the $2q$ complex numbers a_k $(k = 0, \ldots, q-1)$ and q_k $(k = 0, \ldots, q-1)$ that determines these polynomials, and then is $\mathbb{C}^{2q}$. The physical meaning of these parameters will become clear later.

Once the polynomials are constructed, the Higgs fields can be reconstructed with the formula

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \frac{1}{\sqrt{|P|^2 + |Q|^2}} \begin{pmatrix} P \\ Q \end{pmatrix} \exp\left\{\frac{1}{2}u\right\} \tag{3.115}$$

3.6.4 A string like solution in $\mathbb{R}^2$

The case $q = 1$ is easy to study. We will fix the zero of the first component of the Higgs field at the origin (other solutions are a mere translation), and let the remaining parameter that determines the solution ($q_0 \in \mathbb{C}$) arbitrary. This corresponds with the following polynomials

$$P(z) = z = re^{\imath\theta} \tag{3.116a}$$
$$Q(z) = q_0 \tag{3.116b}$$

The form of the Higgs field becomes

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \frac{1}{\sqrt{r^2 + |q_0|^2}} \begin{pmatrix} re^{\imath\theta} \\ q_0 \end{pmatrix} \exp\left\{\frac{1}{2}u(r; |q_0|)\right\} \tag{3.117}$$

This solutions represent a configuration with cylindrical symmetry. The meaning of the parameter q_0 can be obtained by computing the asymptotic form of the solutions. If we parametrise both the Higgs and gauge fields as

$$\phi_a = f_a(r)e^{\imath\theta} \tag{3.118a}$$
$$A_i = \varepsilon_{ij}\partial_j h(r) \tag{3.118b}$$

where $f_a(r)$ and $h(r)$ are real functions. Taking into account that our cylindrically symmetric solution obey

$$\frac{f_2}{f_1} = \frac{q_0}{r} \tag{3.119}$$

the Bogomolny equations for the functions $h(r)$ and $f_1(r)$ are

$$f_1' + \frac{rh' - 1}{r} f_1 = 0 \tag{3.120a}$$
$$h' + rh'' + r\left[f_1^2\left(1 + \frac{q_0^2}{r^2}\right) - 1\right] = 0 \tag{3.120b}$$

and we can obtain the asymptotic behaviour of these fields (see [2, 1])

$$\lim_{r\to\infty} f_1 = 1 - \frac{|q_0|}{2r} + |q_0|^4 \frac{3/8 - 1/|q_0|^2}{r^4} + \mathcal{O}(r^{-6}) \tag{3.121a}$$
$$\lim_{r\to\infty} h(r) = \ln r + \frac{|q_0|^2}{2r^2} - |q_0|^4 \frac{1 - 4/|q_0|^2}{4r^4} + \mathcal{O}(r^{-6}) \tag{3.121b}$$
$$\lim_{r\to\infty} B(r) = 2\frac{|q_0|^2}{r^4} + \mathcal{O}(r^{-6}) \tag{3.121c}$$

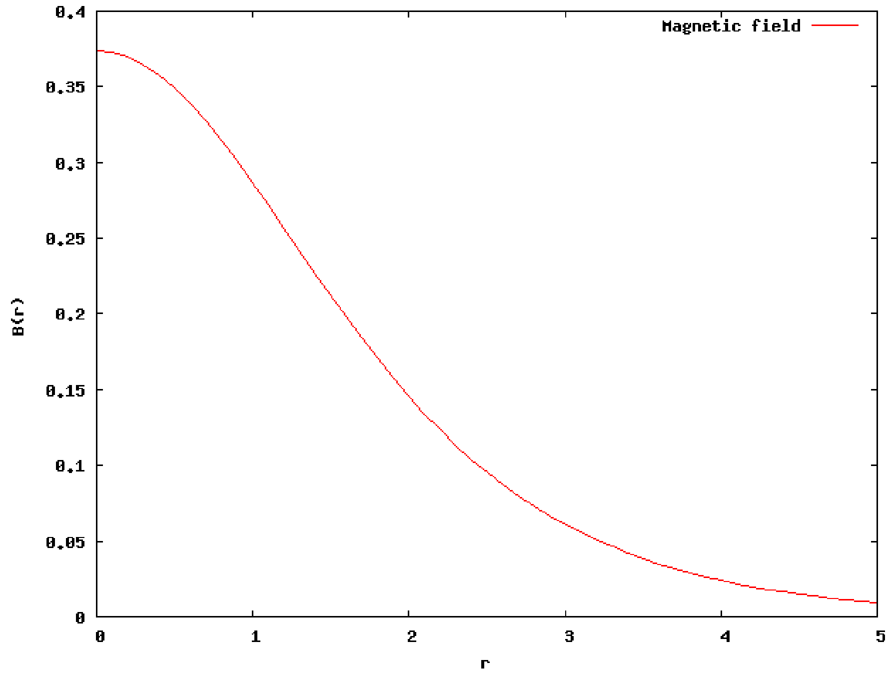


Figure 3.6: Magnetic field of a semilocal string with $|q_0| = 0.5$

This completes the interpretation of these cylindrically symmetric solutions as a vortex string with a width controlled by the parameter $|q_0|$. The profile of the magnetic field of one of these solutions can be seen in the figure (3.6)

In the limit $q_0 \to 0$ we recover the usual Nielsen Olesen vortex string with an exponential decay and $\phi_2 = 0$. In the opposite limit $|q_0| \to \infty$, the "source" term in the equation for the function u (equation (3.111)) is given by

$$\lim_{|q_0|\to\infty} \rho = \frac{4|q_0|^2}{(r^2 + |q_0|^2)^2} \sim 0 \tag{3.122}$$

and then

$$u = 0 \tag{3.123}$$

so the Higgs field lies on the vacuum manifold S^3

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \frac{1}{\sqrt{|z|^2 + |q_0|^2}} \begin{pmatrix} z \\ q_0 \end{pmatrix} \tag{3.124}$$

and this Higgs field modulo gauge transformations defines a map from the plane $\mathbb{R}^2 \sim \mathbb{C}$ into $\mathbb{CP}^1 \sim S^2$. This map is analytic and has degree one, and then is a $\mathbb{CP}^1$ lump. The soliton solutions of the extended abelian Higgs model can be considered as an hybrid between the usual Nielsen Olesen vortices (in the limit $|q_0| \to 0$), and a $\mathbb{CP}^1$ lump.

3.6.5 Multivortex configurations

As we have seen, the moduli space of solutions of the Bogomolny equations with flux number q is given by $\mathbb{C}^{2q}$, thus a general solution of the Bogomolny equations is completely determined by specifying $4q$ real numbers. To give to this numbers a physical meaning, we will write

$$w(z) = \frac{Q(z)}{P(z)} = \sum_{r=1}^{q} \frac{\lambda_r}{z - Z_r} \tag{3.125}$$

the coordinates λ_r and Z_r also parametrise the moduli space (although they are singular when the positions of two different zeroes of ϕ_1 coincide). At least when the vortices are one far from the other, we can interpret the solutions of the Bogomolny equations as q vortices with arbitrary positions, given by Z_r, and arbitrary sizes and orientations in $SU(2)$, given by the modulus and arguments of λ_r.

When the vortices approach one the other this interpretation is not clear at all, there is no a priori reason why the vortices must be located at the points Z_r.

Notice that the "mixed" coordinates Z_r and q_r also parametrise the moduli space. These coordinates will be convenient in the future to compute the metric in the moduli space.

Four

The Abelian Higgs model in the Torus

As we have commented in the introduction of this work, the study of the abelian fields in the Torus has a long history (see for example [27] for a review), and there are several good reasons to do that. First of all the most successful strategy to study the non-perturbative aspects of Yang-Mills theories is probably lattice field theory: it gives a deep understanding of the qualitative behaviour of Quantum Field Theory, and also gives us a way to analyse quantitative results thanks to new methods of computation. Usually when doing numerical computations you have to restrict to a finite size system, and periodic boundary conditions are a usual choice, so want it or not, we have the fields living in a Torus. Second, field configurations in a Torus can be seen as periodic configurations in $\mathbb{R}^n$, in this way a single vortex solution in the torus, for example, can be viewed as an array of vortices in $\mathbb{R}^2$. We can thus study properties of a system with a high density of objects by studing the objects in the torus. Third if our Torus is large enough, localised field configurations in the torus can be seen as configurations belonging to $\mathbb{R}^n$ with a high precision (as the reader will see, we will make use of this property later), in other words, if you take the torus large enough, the only relic of the boundary are the winding numbers (take a look at appendix 5, chapter 7 of [11]). Last, the non trivial topology of the Torus has been used by several people to give a mathematical characterisation of some aspects of the Theory. This was 't Hooft original motivation when he studied non abelian gauge fields living in the four dimensional Euclidean torus to characterise confinement [23, 24].

Here we will only detail Abelian gauge and matter fields in the two dimensional Torus $\mathbb{T}^2$, for other groups or dimensions the reader should check [27] and references therein.

4.1 Fields on the Torus

We will work in a two dimensional torus with Euclidean metric. The length of the sides will be given by l_1 and l_2, and we will choose them orthogonal. The generalisation to other metrics and/or other tilts is possible (see for example [71, Appendix A]), but will not be covered here.

We can see our two dimensional torus as a "periodic" plane, where all the "physical" quantities must be periodic in the two directions. But the non physical (gauge variant) terms need not to be periodic, it is enough for them to be equal modulo a gauge transformation.

$$A_i(x + \hat{l}_a) = A_i(x) + \partial_i T_a(x) \tag{4.1a}$$

$$\phi(x+\hat{l}_a) = e^{iT_a(x)}\phi(x) \tag{4.1b}$$

Where $\hat{l}_1 = (l_1, 0)$ and $\hat{l}_2 = (0, l_2)$. Without possibility of confusion we will write $x + \hat{l}_a$ as $x + l_a$, since it must be clear from the context if we mean the vector $\hat{l}_a$ or the number l_a. For this relation to be consistent, we must have

$$\partial_i \left[T_a(x) + T_b(x+l_a)\right] = \partial_i \left[T_b(x) + T_a(x+l_b)\right] \tag{4.2a}$$

$$e^{i[T_a(x)+T_b(x+l_a)]} = e^{i[T_b(x)+T_a(x+l_b)]} \tag{4.2b}$$

Equations (4.2) imply that the flux that crosses our torus is quantised. The flux is given by:

$$\Phi = \int_0^{l_1}\int_0^{l_2} B\mathrm{d}x_2\mathrm{d}x_1 \tag{4.3}$$

where $B = \partial_1 A_2 - \partial_2 A_1$. Integrating by parts

$$\Phi = \int_0^{l_2} \Delta_1 A_2 \mathrm{d}x_2 - \int_0^{l_1} \Delta_2 A_1 \mathrm{d}x_1 \tag{4.4}$$

with $\Delta_i A_j = A_j(x+l_i) - A_j(x)$. But by equation (4.1a), we have

$$\Delta_i A_j = \partial_j T_i \tag{4.5}$$

So the flux gives

$$\Phi = \Delta_2 T_1 - \Delta_1 T_2 \tag{4.6}$$

that, by equation (4.2), is $2q\pi$

It is easy to find functions $T_i(x)$ that obey the boundary conditions. If we put[1] $T_a(x) = \sum_b \varepsilon_{ab} x_b l_a \frac{f}{2}$, we have that the boundary conditions (4.2) gives us

$$f = \frac{2q\pi}{l_1 l_2} \tag{4.7}$$

that is the average magnetic field that crosses our Torus. The magnetic field B is a gauge invariant quantity, so it must be periodic by definition. This means that we can write:

$$B = \sum_{n,m} b_{nm} e^{inx_1/l_1} e^{imx_2/l_2} \tag{4.8}$$

where n, m are integers. It is trivial to show that *only the constant term* ($n = m = 0$) gives a contribution to the flux:

$$\Phi = \int \mathrm{d}^2 x B = \sum_{n,m} b_{nm} \int_0^{l_1} e^{inx_1/l_1}\mathrm{d}x \int_0^{l_2} e^{imx_2/l_2}\mathrm{d}y = \mathcal{A} b_{00} \tag{4.9}$$

where $\mathcal{A} = l_1 l_2$. If the gauge connection $A_i(x)$ is periodic, the B field will not have any constant term, so to have non zero flux is necessary for A_i to have a linear dependence with the coordinates of the torus. This means that the connection is not periodic, and then it is *not* a one form, but we can "extract" the non-periodic part of A_i, and the rest will be a one form. The non periodic part will give us a constant background magnetic field, that gives the flux,

[1] Note that we do not sum over the index a.

and the other parts will "change" the value of the magnetic field in the points, but without changing the flux (ie. they say *how* the flux is distributed on the torus).

The non periodic part of A_i can be written as:

$$A_i^{(0)} = -\frac{f}{2}\varepsilon_{ij}x_j \tag{4.10}$$

The periodic part is a one form, so we can use Hodge Theorem to write *any* A_i as:

$$A_i = \partial_i g + \varepsilon_{ij}\partial_j h + v_i - \frac{f}{2}\varepsilon_{ij}x_j \tag{4.11}$$

Here g, h are 0-forms in the torus (doubly periodic functions), and v_i are harmonic (that in the Torus are just constants).

The v_i are important, since they give the Polyakov loops. It is easy to see, that even when they are constant you cannot change them by a gauge transformation, simply cause a gauge transformation changes $A \to A + \mathrm{d}g$ with g a *0-form* (therefore periodic), and then $\mathrm{d}g$ cannot be constant. More information about this topic can be found in [34].

4.2 Equations of motion and some solutions

The equations of motion are "local" in the sense that the form of the equations does not depend on the global properties of the space in which they are. This means that the equations of motion are exactly the same as in $\mathbb{R}^2$

$$\partial_i F^{ij} = \frac{\imath}{2}\left[\phi(D^j\phi)^* - \phi^* D^j\phi\right] \tag{4.12a}$$

$$D_i^* D^i\phi = -\frac{\lambda}{2}\phi(|\phi|^2 - 1) \tag{4.12b}$$

the difference resides in that now, the fields lives in the Torus. This helps us, and at first sight it could seem a trivial task, because simply putting:

$$\phi = 0; \quad B(x) = \text{constant} = \frac{2q\pi}{l_1 l_2} \tag{4.13}$$

the equations of motion are satisfied, and of course, they are solutions of finite energy (we will call this solution the planar solution). This may make us think that we have the solutions of the equations of motion in a torus. The thing is that these are not the only solutions that exist in a torus. It is straightforward to calculate the energy of these solutions:

$$E = \frac{\Phi}{\mathcal{A}} + \frac{\lambda}{8}\mathcal{A} \tag{4.14}$$

The energy of these solutions grows with the area for a big torus, and then is very big for a very big torus (this is clearly the reason of why they are not solutions of finite energy of the equations of motion in $\mathbb{R}^2$). If there exist other solutions whose energy is more or less independent of the size of the torus (i.e. "localised" solutions, with the asymptotic behaviour predicted by NO), these solutions will have less energy for a big torus.

We can calculate the area $\mathcal{A}$ of our torus to make the energy minimum, and we will obtain

$$\mathcal{A} = 2\Phi\sqrt{\lambda} \tag{4.15}$$

For this value of $\mathcal{A}$, the energy is given by

$$\frac{\sqrt{\lambda}}{4}\Phi\left(\frac{1}{\lambda}+\lambda\right) \tag{4.16}$$

In fact it is easy to see that these solutions satisfy the Bogomolny equations when the area of the torus is just $4q\pi$ (for other values of the area, the solutions satisfy the equations of motion, but not the Bogomolny equations), and then we know that the energy must be $q\pi$. In this case our planar solution saturate the Bogomolny bound, and then these solutions are of minimum energy[2].

Later we will use this solution as a first step for constructing solutions for other torus sizes.

4.3 The Bogomolny equations on the torus

Here we will study the Bogomolny equations and in some sense, "decouple" the two equations. As with the equations of motion, the Bogomolny equations are "local" and then are exactly the same for the Torus than for $\mathbb{R}^2$, also the Bogomolny bound (3.57a) is exactly the same. But for the Torus, there are some differences, the most notable one was first noted by Bradlow [57] in a more general context: the Bogomolny equations impose some constraints in our torus sizes. Integrating the second Bogomolny equation (3.57c), we have:

$$\int |B|\mathrm{d}^2x = 2|q|\pi = \frac{1}{2}\int \mathrm{d}^2x\{1-|\phi|^2\} \tag{4.17}$$

So we have:

$$\mathcal{A} - 4|q|\pi = \int \mathrm{d}^2x|\phi|^2 \tag{4.18}$$

If the area of the torus $\mathcal{A}$ is less than $4|q|\pi$ we cannot find solutions to the Bogomolny equations. As we saw before there are solutions of the equations of motion, but they do not saturate the Bogomolny bound. We will call the $\mathcal{A} = 4|q|\pi$ limit the *Bradlow limit*.

We are going to manipulate the Bogomolny equations[3]:

$$(D_1 + \imath D_2)\phi = 0 \tag{4.19a}$$

$$B = \frac{1}{2}(1-|\phi|^2) \tag{4.19b}$$

It will be very useful to use complex notation:

$$z = \frac{x_1 + \imath x_2}{2}; \qquad \overline{z} = \frac{x_1 - \imath x_2}{2} \tag{4.20a}$$

$$\partial = \partial_1 - \imath\partial_2; \qquad \overline{\partial} = \partial_1 + \imath\partial_2 \tag{4.20b}$$

$$\partial_1 = \frac{1}{2}(\partial + \overline{\partial}); \qquad \partial_2 = \frac{1}{2\imath}(\partial - \overline{\partial}) \tag{4.20c}$$

[2] This solution is very similar to the t'Hooft solutions of self dual $SU(2)$ gauge fields in the $4D$ torus that we have seen in the introduction, that only exist for certain torus sizes.

[3] Note that we choose the positive flux version of the equations. This clearly can always be done, and is not a restriction.

The notation is chosen so that $\partial z = \overline{\partial}\overline{z} = 1$, and we will indicate complex conjugate with a line over the quantity. In general, things with an upper index will use the same notation that coordinates, and things with a lower index will use the same notation that the partial derivatives. Now we can rewrite the Bogomolny equations with this notation:

$$(\overline{\partial} - \imath\overline{A})\phi = 0 \tag{4.21a}$$

$$\frac{\imath}{2}\left(\overline{\partial}A - \partial\overline{A}\right) = \frac{1}{2}(1 - |\phi|^2) \tag{4.21b}$$

(Since A_i has a lower index $A = A_1 - \imath A_2$, and $\overline{A} = A_1 + \imath A_2$). Using the Hodge theorem, we can write any A as:

$$A = -\imath\partial\overline{H} + v - \imath f\overline{z} \tag{4.22}$$

Where $H = h + \imath g$ (see Eq.(4.11)), and $v = v_1 - \imath v_2$.The Bogomolny equations become

$$(\overline{\partial} + fz)\phi = (\imath\overline{v} - \overline{\partial}h)\phi \tag{4.23a}$$

$$\partial\overline{\partial}h = \frac{1}{2}(1 - 2f - |\phi|^2) \tag{4.23b}$$

We will parametrise the Higgs field

$$\phi = \sqrt{1-2f}e^{-H}\chi \tag{4.24}$$

and finally the Bogomolny equations are reduced to:

$$(\overline{\partial} + fz)\chi = \imath\overline{v}\chi \tag{4.25a}$$

$$\partial\overline{\partial}h = \frac{1-2f}{2}(1 - e^{-2h}|\chi|^2) \tag{4.25b}$$

In these equations χ have the same boundary conditions as ϕ, since H is periodic, and there is where all the information about the Torus resides. Note that the first equation is an equation only for the function χ. Once this equation is solved, and χ obtained, we can plug it in the second equation to obtain h. It is also important to note that g (the imaginary part of our function H) does not appear here, because it parametrises the gauge transformations.

4.3.1 The moduli space of solutions

A natural question that arises when one faces these equations is: How many solutions are there to the Bogomolny equations? How can we parametrise these solutions? We know (Section (3.5)) that in the case of $\mathbb{R}^2$ the moduli space has dimension $2|q|$, and that these $2|q|$ parameters can be interpreted as the positions of the $|q|$ zeros of the Higgs field (counted with multiplicity). Here we will prove a similar case for the Torus. We will use the results of the appendix B about the theta functions and the theta functions with characteristics.

To begin, we will assume that the solution of the second Bogomolny equation (4.25b) is unique given χ (we will need to wait until the next section to prove this). First of all we will study the meaning of the constant v. In order to do that, we will call $\chi^{(v)}$ a solution of (4.25a), now it is straightforward to prove that the function χ defined as

$$\chi = \mathcal{T}_v\left[\chi^{(v)}(z,\overline{z})\right] \equiv e^{-\frac{|v|^2}{4f}} e^{\frac{-\imath}{2}(vz+\overline{vz})}\chi^{(v)}\left(z + \frac{\imath\overline{v}}{2f}, \overline{z} - \frac{\imath v}{2f}\right) \tag{4.26}$$

is a solution of (3.5) but for the value $v = 0$. Note that the operator $\mathcal{T}_v$ is a translation modulo a rescaling and a gauge transformation. In particular this means that $\mathcal{T}_v$ only can "move" the position of all the zeros of the field over which it acts, but cannot neither change their relative position nor the number of zeros. Now making a translation also to the function h, we can associate a solution of the Bogomolny equations for any value of v to a unique solution of the Bogomolny equations for $v = 0$. These means that v can be taken as a coordinate of the moduli space, that can be interpreted as the position of the centre of mass of the zeros of the Higgs field (we will give more details about that later).

Second we will study the first Bogomolny equation for the special value $v = 0$, since we can always construct a solution with arbitrary v using the operator $(\mathcal{T}_v)^{-1}$. This equation reads

$$(\overline{\partial} + fz)\chi = 0 \tag{4.27}$$

as is explained in appendix C, this equation can be seen as the equation defining the ground state of a $|q|$ dimensional harmonic oscillator, and has $|q|$ linearly independent solutions, that we will label χ_i, and can be expressed in terms of the Theta functions with characteristics (for a definition of these functions see the appendix B)

$$\chi_i = e^{fz(z-\overline{z})}\vartheta \left[\begin{array}{c} i/q \\ 0 \end{array} \right] \left(\frac{2q\pi z}{l_1} | iq\frac{l_2}{l_1} \right) \tag{4.28}$$

Given the linear character of the equation (4.27), its general solution is

$$\chi = \sum_{i=1}^{q} c_i \chi_i \tag{4.29}$$

where c_i are arbitrary complex constants. We note here (more details in appendix C) that the functions χ_i are orthonormal with respect to the scalar product defined by

$$\langle \eta_1 | \eta_2 \rangle = \frac{1}{\mathcal{A}} \int \overline{\eta_1} \eta_2 \, d^2x \tag{4.30}$$

In principle, one might be led to conclude that the set of $|q|$ complex constants c_i, together with the coordinate v provide coordinates of the moduli space of solutions. However, notice that the Higgs and gauge field are invariant (up to a global gauge transformation) under the transformation:

$$c_i \longrightarrow \kappa c_i; \qquad h \longrightarrow h + log(|\kappa|) \tag{4.31}$$

where κ is an arbitrary complex number. Hence, for fixed v the space of solutions becomes equivalent to the complex projective space $\mathbb{CP}^{q-1}$ and the coefficients c_i can be regarded as homogeneous coordinates in this manifold. This completes the structure of the moduli space as a fibre bundle with fibre $\mathbb{CP}^{q-1}$, and bundle parametrised by the coordinate v. In fact this is a particular case of more general results for vortices in compact manifolds developed in [72].

So the moduli space has the structure of a fibre bundle that can be parametrised by the coordinates

$$\mathcal{B}_1 = \{v, \overline{v}, c_i, \overline{c}_i\}; \; (i = 1, \ldots, q) \tag{4.32}$$

where v parametrise the base space, and the c_i are homogeneous coordinates of the fibre, that is $\mathbb{CP}^{q-1}$. There are several ways to obtain inhomogeneous coordinates. A simple choice is to divide all the c_i by one of them (say c_1). Now the $q-1$ coordinates $Z_i = c_{i+1}/c_1$ $(i = 1, \ldots, q-1)$

gives inhomogeneous coordinates of $\mathbb{CP}^{q-1}$in the region where $c_1 \neq 0$. For the special case $q = 1$ the $\mathbb{CP}^1 \approx S_2$, and there exist more natural possibilities to parametrise the moduli space (this special case, will be studied in detail later). It is natural to ask how these coordinates relate to the more physical parametrisation of the moduli in terms of the position of the zeros of the Higgs field. To answer this, we first note that the function

$$\chi = e^{fz(z-\overline{z})} \prod_{i=1}^{q} \vartheta_3 \left(\frac{2\pi(z + Z_{\mathbb{T}^2} - w_i)}{l_1} |i \frac{l_2}{l_1} \right) \tag{4.33}$$

where $Z_{\mathbb{T}^2} = \frac{1}{4}(l_1 + \imath l_2)$ and w_i are $|q|$ arbitrary complex numbers is also a solution of (4.27) with the correct boundary conditions, provided that the complex numbers w_i obey

$$\sum_{i=1}^{q} w_i = q Z_{\mathbb{T}^2} \tag{4.34}$$

From the properties of the Theta functions (details in appendix B), this function has exactly $|q|$ zeros (counted with multiplicity), located at the points w_i. From the functions χ_i defined in (4.28) we can construct a function that has at least $q-1$ zeros located in the positions $w_i\,(i = 1, \ldots, q-1)$, choosing the following $c's$

$$c_s = \varepsilon_{s i_1 \cdots i_{q-1}} \chi_{i_1}(w_1) \cdots \chi_{i_{q-1}}(w_{q-1}) \tag{4.35}$$

From theorem (B.4.1), the functions $\sum_s c_s \chi_s$ and the function χ defined in (4.33) must be proportional, then they have the $|q|$ zeros located in the same positions. This gives the relation between the homogeneous parametrisation of the fibre in terms of the coordinates c_i, and the more physical coordinates w_i (note that because of (4.34) not all w_i are independent). Since the condition (4.34) can be interpreted as the centre of mass of the zeros of the Higgs field being in the centre of the Torus, once we apply the operator $(\mathcal{T}_v)^{-1}$ to the function χ, the centre of mass of the zeros of χ will be located in

$$Z_{\mathbb{T}^2} + \imath \frac{\overline{v}}{2f} \tag{4.36}$$

This complete the interpretation of v as a parameter that gives the centre of mass of the positions of the zeros of the Higgs field.

4.4 The Extended Abelian Higgs Model on the torus

Here we will work with the extended abelian Higgs model with a symmetry group $SU(N)_{\text{global}} \times SU(N')_{\text{global}} \times U(1)_{\text{local}}$, with Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (D_\mu \phi_A)^* (D^\mu \phi_B) - \frac{\lambda}{8} \left(|\phi_a|^2 - |\phi_r|^2 - 1 \right)^2 \tag{4.37}$$

The general notation used in this section is the same as in the section 3.6. The Bogomolny equations for this extended abelian Higgs model were deduced in the section 3.6, and are given by

$$D_- \phi_a = (D_1 + \imath D_2) \phi_a = 0 \qquad (a = 1, \ldots, N) \tag{4.38a}$$

$$D_+ \phi_r = (D_1 - \imath D_2) \phi_r = 0 \qquad (r = N+1, \ldots, M) \tag{4.38b}$$

$$B = \frac{1}{2} \left(1 - \phi_A^* \eta_{AB} \phi_B \right) \tag{4.38c}$$

here we will proof that the only solution to the equation (4.38b) is given by $\phi_r = 0$, just as in the case of vortices living in $\mathbb{R}^2$ (see section 3.6.1). To proof that we will use again the Hodge decomposition theorem to write, with our complex notation

$$A = -\imath\partial\overline{H} + v - \imath f\overline{z} \tag{4.39}$$

And again, we will write

$$\begin{aligned} \phi_a &= \sqrt{1-2f}\, e^{-H}\chi_a^{(v)} \\ \phi_r &= \sqrt{1-2f}\, e^{\overline{H}}\chi_r^{(v)} \end{aligned} \tag{4.40}$$

(4.41)

Thus the Bogomolny equations are reduced to

$$(\overline{\partial} + fz)\chi_a^{(v)} = \imath\overline{v}\chi_a^{(v)} \qquad (a = 1,\ldots,N) \tag{4.42a}$$

$$(\partial - f\overline{z})\chi_r^{(v)} = \imath v\chi_a^{(v)} \qquad (a = 1,\ldots,N) \tag{4.42b}$$

$$\overline{\partial}\partial h = \frac{1-2f}{2}\left(1 - e^{-2h}|\chi_a^{(v)}|^2 + e^{2h}|\chi_r^{(v)}|^2\right) \tag{4.42c}$$

The solutions of the first two Bogomolny equations for a value $v \neq 0$ are related to the solutions of these equations for $v = 0$ via the operator $\mathcal{T}_v$ defined in equation (4.26)

$$\chi_A^{(0)} = \mathcal{T}_v\left[\chi_A^{(v)}(z,\overline{z})\right] \equiv e^{-\frac{|v|^2}{4f}} e^{\frac{-\imath}{2}(vz+\overline{v}\overline{z})}\chi_A^{(v)}\left(z + \frac{\imath\overline{v}}{2f}, \overline{z} - \frac{\imath v}{2f}\right) \tag{4.43}$$

since this result can be extended to the last Bogomolny equation by making a translation to the function h, it is enough to find the solutions of equations (4.42a) and (4.42b) for the special case $v = 0$. But as the reader can see in the appendix C, these equations can be interpreted in terms of creation and annihilation operators acting on the Hilbert space of quasi periodic functions. This means in particular that the general solution to the two first Bogomolny equations depends on Nq complex constants $c_s^{(a)}; (a = 1,\ldots,N; s = 1,\ldots,q)$ in the following way

$$\chi_a^{(0)} = \sum_s c_s^{(a)} \mid 0, s\rangle \tag{4.44}$$

$$\chi_r^{(0)} = 0 \tag{4.45}$$

where the functions $\mid 0, s\rangle$ are defined in equation (C.22), and can be written in terms of the Theta functions with characteristics defined in appendix B (the details of all this computations can be followed from first principles in appendix C)

$$\mid 0, s\rangle = \left(\frac{2ql_2}{l_1}\right)^{1/4} e^{\imath\frac{f}{2}x_1x_2} e^{-\frac{f}{2}x_2^2}\vartheta\left[\begin{array}{c} s/q \\ 0 \end{array}\right]\left(\frac{2\pi qz}{l_1}|\imath q\frac{l_2}{l_1}\right) \tag{4.46}$$

4.4.1 The moduli space

The analysis previously done, allows us to write the form of the moduli space. As we have seen, the parameter v can be associated with the position of the centre of mass of the zeros of *all* the Higgs fields, and is then associated with the invariance under translations. The solutions for any value of v can be related, through the operator $\mathcal{T}_v$ to the solution for the special value $v = 0$, and this result can be extended to the last Bogomolny equation by making a translation to the function h. This means that the moduli space has a fibre bundle structure with base space parametrised by the coordinate v. To know the structure of the fibre it is enough to analyse the solutions of the Bogomolny equations for the special value $v = 0$.

For the moment we will assume that the last Bogomolny equation has a unique solution for some given complex numbers $c_s^{(a)}$. As we have seen the (normalised) solution to the first Bogomolny is unique given Nq complex numbers $c_s^{(a)}$, and then all the moduli resides there. As in the non extended abelian Higgs model, all the Bogomolny equations are invariant under the transformation

$$\begin{aligned} c_i^{(a)} &\longrightarrow \kappa c_i^{(a)} \\ h &\longrightarrow h + log(|\kappa|) \end{aligned} \tag{4.47}$$

with κ an arbitrary complex constant. Thus the fibre has the structure of the complex projective space $\mathbb{CP}^{Nq-1}$, with $c_i^{(a)}$ being homogeneous coordinates. Using the previous relations we can relate the values of the homogeneous coordinates with the positions of the zeros of the Higgs field (see equation (4.35)). The odd thing is that being v unique, all the components of the Higgs field must have the same position of the centre of mass. This allows to simplify the Bogomolny equations in the following manner: taking $v = 0$, the last Bogomolny equation reads

$$\overline{\partial}\partial h = \frac{1-2f}{2}\left(1 - e^{-2h}|\chi_a^{(0)}|^2\right) \tag{4.48}$$

and using the results of the appendix C we can write

$$|\chi_a^{(0)}|^2 = \left(\sum_{a=1}^{N} \overline{c}_s^{(a)} c_{s'}^{(a)}\right) \langle 0, s|0, s'\rangle \tag{4.49}$$

we see that the magnetic field only "sees" the N scalar fields through the hermitian $q \times q$ matrix

$$M_{ss'} = \sum_{a=1}^{N} \overline{c}_s^{(a)} c_{s'}^{(a)} \tag{4.50}$$

Although in principle the moduli space depends on q vectors of N components, we have seen here that only the $q(q+1)/2$ scalar products among them enter in the vortex equation. Moreover the freedom that we have to multiply all the complex numbers $c_i^{(a)}$ by a complex constant can be used to fix the value of one component of the matrix $M_{ss'}$, in particular we will use

$$M_{11} = 1 \tag{4.51}$$

The coordinates $c_i^{(a)}$ that parametrise the moduli space, can be seen as a $N \times q$ matrix

$$c_i^{(a)} = \begin{pmatrix} c_1^{(1)} & c_2^{(1)} & \dots & c_q^{(1)} \\ c_1^{(2)} & c_2^{(2)} & \dots & c_q^{(2)} \\ \vdots & \vdots & \vdots & \vdots \\ c_1^{(N)} & c_2^{(N)} & \dots & c_2^{(N)} \end{pmatrix} \tag{4.52}$$

An transformation $\Omega \in SU(N)$ acts on this matrix through the left

$$c \longrightarrow \Omega c \tag{4.53}$$

and the $q \times q$ matrix M that enters in the vortex equation is defined by the equation

$$M = c^+ c \tag{4.54}$$

it is straightforward to prove that M is invariant under an $SU(N)$ transformation of the matrix c, just as we have seen.

Having characterised the moduli for the general case, we will concentrate in the case $N = 2$.

4.4.2 The case $N = 2$

The case $N = 2$ is the one that has been investigated in depth for the case of vortices living in $\mathbb{R}^2$. As we have seen in section 3.6.3, after fixing the $SU(2)$ global symmetry specifying the behaviour of the Higgs fields at infinity, there exist a unique solution given $2q$ complex numbers. Thus the complete moduli space is given by $\mathbb{C}^{2q}$ plus a $SU(2)$ rotation, having in total $4q + 3$ parameters that we need to completely specify the form of the solution. The magnetic filed, being an $SU(2)$ invariant quantity, only depends on $4q$ real parameters.

The case of the torus is very different. Here the complete moduli space depends on the $2q - 1$ complex coordinates of the fibre plus the 2 coordinates of the base space, making a total of $4q$ real parameters that we need to completely specify the moduli. The magnetic field depends only on the parameters that enter in the hermitian matrix M.

Now we will examine the implications of these properties on the form of the solution for some representative examples. In order to do that it will be very useful to make the *QR decomposition* of the matrix $c_i^{(a)}$. This means that for $q \leq 2$ we can write *any* $c_i^{(a)}$ as the product of an unitary $N \times q$ matrix and a square $q \times q$ positive definite upper triangular matrix

$$c = \Omega R \tag{4.55}$$

where

$$\Omega^+ \Omega = \mathbb{I} \tag{4.56}$$

All the information that we need to solve the vortex equation resides in the matrix R, because we can easily check that

$$M = R^+ R \tag{4.57}$$

For the case $q = 1$ the matrix M is simply a number, and with the freedom that we have to multiply it by any complex number, we can choose it equal to 1. This is a particular case of what we said in the previous section: the $q = 1$ solutions of the extended abelian Higgs model in the torus are rather trivial since the magnetic field is exactly the same as the one of the usual abelian Higgs model.

The case $q = 2$ is more interesting, using the remaining freedom that we have, we can choose the matrix $c_i^{(a)}$ in the following way

$$c = \begin{pmatrix} 1 & b \\ 0 & c \end{pmatrix} \tag{4.58}$$

where b is a complex number, and c is real. The form of the matrix M that enters in the vortex equation is

$$M = c^{+}c = \begin{pmatrix} 1 & cb \\ c\bar{b} & c^2 \end{pmatrix} \tag{4.59}$$

and then the moduli space for this case depends on three parameters (we have to remember in this point that we have fixed the position of the centre of mass of the zeros of all the Higgs fields). The meaning of these parameters is unclear at this point, and we will have to wait until we actually find the solutions of these equations to see what they mean.

Five

Solutions of the Bogomolny equations in the Torus

5.1 The Bradlow parameter expansion

Since we have solved the first Bogomolny equation, and obtained χ as a function of the coordinates of the moduli space, we only have to solve the so called "vortex equation"

$$\partial\overline{\partial}h = \frac{1-2f}{2}(1-e^{-2h}|\chi|^2) \tag{5.1}$$

for a periodic function h, given a (complicated) function χ, that we will assume expanded in the homogeneous (orthonormal) basis $\chi = \sum_i c_i \chi_i$. The Bradlow limit $\mathcal{A} = 4q\pi$, can be recovered here simply putting

$$\epsilon \equiv 1 - 2f = 1 - \frac{4q\pi}{\mathcal{A}} = 0 \tag{5.2}$$

For this special case the Higgs field given by

$$\phi = \sqrt{\epsilon}e^{-H}\chi \tag{5.3}$$

is zero, and the solution to the second Bogomolny equation is given by $h = cte$ (later we will show how to determine the constant). We will call ϵ the Bradlow parameter. This parameter interpolates between the minimum area for which the Bogomolny equations have solutions ($\epsilon = 0$), and the limit case of infinite are (vortices in the plane), given by $\epsilon = 1$.

When ϵ is close to zero, near to the Bradlow limit, we expect for the solution to the vortex equation to be close to the planar solutions, and this is the origin of the Bradlow parameter expansion. Since h is periodic, we will expand it in the Fourier basis

$$h = \sum_{\vec{n}} h(\vec{n}) e^{2\pi\imath n_1 \frac{x_1}{l_1}} e^{2\pi\imath n_2 \frac{x_2}{l_2}} \tag{5.4}$$

where $\vec{n} = (n_1, n_2)$. We will expand the Fourier modes $h(\vec{n})$ as a power series in the Bradlow parameter ϵ

$$h(\vec{n}) = \sum_k h^{(k)}(\vec{n})\epsilon^k \tag{5.5}$$

and try to solve the vortex equation order by order in ϵ. As an input, we have the Fourier modes of $|\chi|^2$. Since in the Bradlow limit $\epsilon = 0$ the function h is also zero, we have the zero order of the series solved

$$h^{(0)}(\vec{n}) = 0 \quad \vec{n} \neq 0 \tag{5.6}$$

To determine $h^{(0)}(\vec{0})$, we will need the *next* order of the vortex equation, that reads

$$\partial\overline{\partial}h = \frac{\epsilon}{2}\left(1 - e^{-2h^{(0)}(0)}|\chi|^2\right) \tag{5.7}$$

Integrating both sides in the Torus, we get

$$h^{(0)}(0) = \ln(\|c\|) \tag{5.8}$$

where $\|c\|^2 = \sum_i |c_i|^2$.

In general, the l.h.s. of the vortex equation can be written as

$$-(1-\epsilon)\xi(\tau, n_1, n_2)\sum_k h^{(k)}(\vec{n})\epsilon^k \tag{5.9}$$

where

$$\xi(\tau, n_1, n_2) = \frac{\pi\tau}{q}\left[n_1^2 + \frac{n_2^2}{\tau^2}\right] \tag{5.10}$$

and $\tau = l_2/l_1$. We note that ξ vanishes for $\vec{n} = 0$.

To treat the r.h.s. we first expand it in powers of ϵ. To order ϵ the coefficient is given by $(1 - e^{-2h^{(0)}(0)}|\chi|^2)/2$. The coefficient of order ϵ^N (for $N > 1$) is given by

$$-\frac{1}{2}\sum_{k=0}^{N-1}\mathtt{A}_{N-k}\left[\frac{(-2)^k}{k!}\sum_{\substack{i_1,\ldots,i_{N-1}\\ /\sum_s s i_s = k}}\binom{\sum_s i_s}{i_1,\ldots,i_{N-1}}h^{(1)^{i_1}}(\vec{n}) * \cdots * h^{(N-1)^{i_{N-1}}}(\vec{n})\right]|\chi|^2 \tag{5.11}$$

where the symbol $*$ means that we have to make the *convolution* of the Fourier modes, and the constants $\mathtt{A}_k$ are defined by

$$\epsilon e^{-2h(0)} = \sum_k \mathtt{A}_k\epsilon^k \tag{5.12}$$

(in particular this means that $\mathtt{A}_1 = 1/|c|^2$). Now, using the Fourier coefficients of $|\chi|^2$ (see the appendix C), we can obtain the Fourier coefficients of the expression above by applying a series of convolutions. We will label these Fourier modes with the symbol $\mathcal{F}^{(N)}(\vec{n})$. These are functions of $\mathtt{A}_i(i = 1, \ldots, N)$, and $h^{(i)}(i = 1, \ldots, N-1)$. In order for the equation to have a solution it is necessary that $\mathcal{F}^{(N)}(0) = 0$. This can be regarded as an equation for $\mathtt{A}_N$. It allows to determine this coefficient uniquely in terms of $\mathtt{A}_i(i = 1, \ldots, N-1)$, and $h^{(i)}(i = 1, \ldots, N-1)$.

Finally, the equation allows one to obtain the Fourier coefficients $h^{(N)}(\vec{n})$ to order ϵ^N as follows:

$$h^{(N)}(\vec{n}) = -\frac{\mathcal{F}^{(N)}(\vec{n})}{\xi(\tau, n_1, n_2)} + h^{(N-1)}(\vec{n}) \qquad \vec{n} \neq 0 \tag{5.13}$$

and the zero mode $h^{(N)}(0)$ can be obtained from the constants $\mathtt{A}_i$ $(i = 1, \dots, N)$. This equation defines a recurrence which, starting from $h^{(0)}(\vec{n}) = 0$, enables the determination of the coefficients $h^{(N)}(\vec{n})$ uniquely.

We emphasise that the previous procedure gives rise to a unique solution h, hence proving that the solution to the second Bogomolny equation is unique given χ.

5.1.1 First order computation

We have previously computed the function h to zero order, that has only a constant term, given by

$$h^{(0)}(\vec{n}) = \begin{cases} 0 & \vec{n} \neq 0 \\ \ln |c| & \vec{n} = 0 \end{cases} \tag{5.14}$$

to obtain the function up to first order, we will need the Fourier coefficients of the product of two elements of the basis

$$L^{ij}(\vec{n}) = \frac{1}{\mathcal{A}} \int_{\mathbb{T}^2} d^2x\, \overline{\chi}_i e^{-2\pi\imath\left(n_1 \frac{x_1}{l_1} + n_2 \frac{x_2}{l_2}\right)} \chi_j \tag{5.15}$$

this Fourier coefficients can be explicitly obtained with the results of the appendix C. We will re-define the zero Fourier mode of this coefficients to be zero (i.e. we put by hand $L^{ij}(0) = 0$). $L(\vec{n})$ without the indices omitted stands for the contraction

$$L(\vec{n}) = \overline{c}_i c_j L^{ij}(\vec{n}) \tag{5.16}$$

Everything will be written in terms of convolutions of these Fourier modes. In particular we will need

$$S_{ijkl} = \sum_{\vec{n}} \frac{L^{ij}(\vec{n}) L^{kl}(-\vec{n})}{\xi(\vec{n})} \tag{5.17}$$

where $\xi(\vec{n})$ is a short name for $\xi(\tau, n_1, n_2)$ defined in equation (5.10). S without indices stands for the contraction

$$S = \overline{c}_i c_j \overline{c}_k c_l S_{ijkl} \tag{5.18}$$

With this definitions in mind it is easy to obtain h up to first order

$$h^{(1)}(\vec{n}) = \begin{cases} \frac{L'(\vec{n})}{2|c|^2 \xi(\vec{n})} & \vec{n} \neq 0 \\ -\frac{S}{2|c|^4} & \vec{n} = 0 \end{cases} \tag{5.19}$$

5.1.2 second order computation

The second order computation only requires more work, but can also be done by hand. Here the result is

$$h^{(2)}(\vec{n}) = \begin{cases} \frac{1}{2|c|^2} \left(\frac{S}{|c|^4} + 1 \right) \frac{L(\vec{n})}{\xi(\vec{n})} - \frac{1}{2|c|^4 \xi(\vec{n})} \sum_{\vec{k}} \frac{L(\vec{k}) L(\vec{n}-\vec{k})}{\xi(\vec{k})} & \vec{n} \neq 0 \\ \frac{1}{|c|^4} \sum_{\vec{n},\vec{k}} \frac{L(\vec{k}) L(\vec{n}-\vec{k}) L(\vec{-n})}{\xi(\vec{k}) \xi(\vec{n}-\vec{k})} - \frac{S^2}{2|c|^6} - \frac{S}{2|c|^2} & \vec{n} = 0 \end{cases} \tag{5.20}$$

5.1.3 Higher orders in the series

To compute analytically higher orders in these series will be in general very costly, so we have to switch to numerical methods. With the help of a computer we cannot make the infinite sums that a convolution of two Fourier series involves. But if we take a look to the Fourier coefficients of $|\chi|^2$

$$L(\vec{n}) = e^{-\frac{\xi(\vec{n})}{2}} e^{i\pi\frac{n_1 n_2}{q}} \left(\sum_s c_s c^*_{s+n_1} e^{2\pi i \frac{s n_2}{q}} \right) \tag{5.21}$$

we see that they decrease exponentially fast with $\vec{n}^2$. This means that, as everything is build from convolutions of these Fourier modes, truncating the infinite sums to a small number of modes will allow the determination of the desired quantities up to machine precision with great efficiency.

In fact we will able to determine 51 orders of the expansion up to machine precision, as we will see in the following section.

5.2 Numerical Computation

In this section we will show the effectiveness of the method presented in the previous section, by applying it to two explicit examples. The first one is the standard $q = 1$ case, which *should* converge towards the usual Nielsen-Olesen vortex with cylindrical symmetry when the Torus is big enough. An extensive analysis of convergence of the series for different volumes will be presented. The second example is a $q = 2$ case which has no remnant continuous spatial symmetry. This is important since many alternative methods are specific to cylindrical symmetry.

5.2.1 $q = 1$

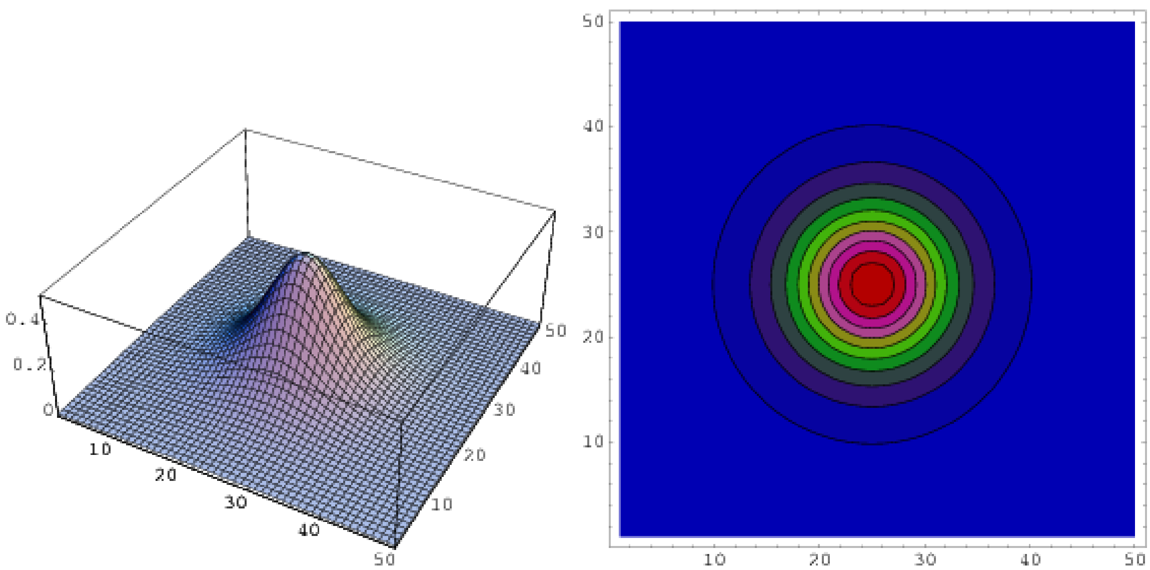
We have applied the machinery explained in the previous section to the unit flux $q = 1$ and unit aspect ratio $\tau = l_2/l_1 = 1$ case. The solution is unique up to translations. We have obtained the coefficients of the expansion $h^{(k)}_{n_1 n_2}$ up to order $k = 51$. Convolutions were performed by truncating the sums to Fourier modes in the range $|n_i| \leq n_{\max}$. This was a implemented using a `FORTRAN 90` program running in a standard PC. Results up to $k = 51$ required computation times around 100 hours.

We have, first of all, analysed the effect of truncation in the number $n_{\max}$ of Fourier modes used in convolutions on the value of the coefficients. We explored the cases $n_{\max} = 10, 14$ and 20. We estimate the relative difference (difference over sum) in the coefficient $h^{(k)}_{n_1 n_2}$ obtained from two different truncations 10-14 or 14-20. The difference is attributed to an error in the value for the smaller $n_{\max}$. The differences increase with n_i and with k. However, even for the highest order ($k = 51$) the relative difference between the results with $n_{\max} = 14$ and 20 is of the order of machine precision $10^{-16,-17}$ for all modes having $n_i < 7$. For higher modes it is to be expected that the error is mostly due to an error in the values for $n_{\max} = 14$, so the modes for $n_{\max} = 20$ remain probably within machine precision for higher n_i. To analyse this we used the comparison 10-14 to try to understand the dependence of errors with n_i, k and $n_{\max}$. It is to be expected that the error is proportional to (some power) of the size of the neglected terms $h^{(k)}_{n_{\max} n_{\max}}$. This fits nicely with the observed dependence of the relative difference

$$-3(1)-\frac{1270(30)}{k}+\max(n_1,n_2)\left(-0.15(6)+\frac{54(1)}{k}\right) \tag{5.22}$$

Our interpretation of the source of the errors suggest that the coefficients multiplying $\frac{1}{k}$ are proportional to $n^2_{\max}$. This allows us to scale the results to $n_{\max}=14$. Indeed, our data on the difference 14-20 is consistent with this interpretation, although there are few values of k and n_i to allow an analysis by itself. On the basis of these results, we conclude that, even at $k=51$, the coefficients obtained for $n_{\max}=20$ are correct up to machine (double) precision for $n_i\leq 14-15$.

Once the coefficients are obtained, one can reconstruct the Fourier modes of h, the magnetic field B and the modulus of the Higgs field $|\phi|$ for arbitrary torus areas $\mathcal{A}=l_1l_2=\frac{4\pi}{1-\epsilon}$. Obviously, the precision of the truncated series depends on the value of ϵ, decreasing as ϵ increases up to the infinite area case of $\epsilon=1$. Even for relatively large areas the shape of the reconstructed function looks qualitatively quite good. See for example the figure (5.1), where we display the magnetic field for $\epsilon=0.9$, obtained from the Fourier modes ($|n_i|\leq 20$) computed with 51 orders in the expansion. To go beyond the qualitative level and estimate the



(a) Shape of the magnetic field $B(x)$ obtained for $\epsilon=0.9$ with 51 orders in the expansion.

(b) Contour plot of $B(x)$.

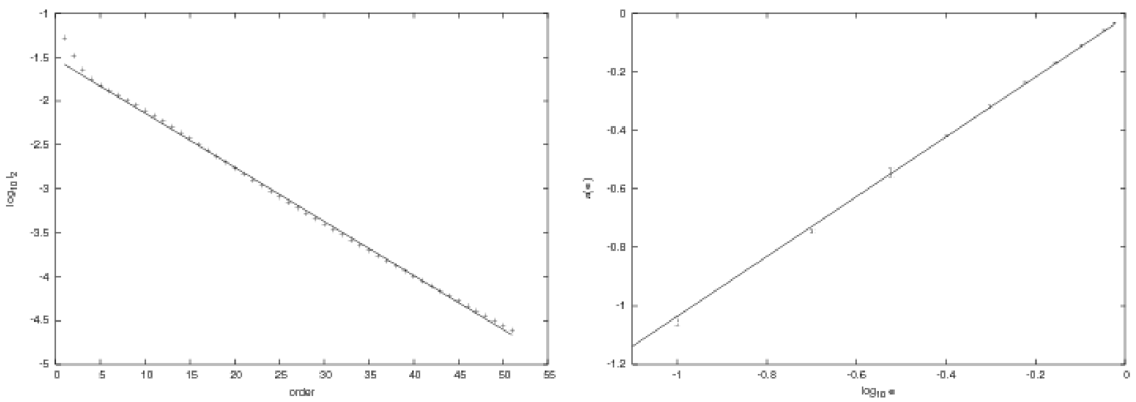
Figure 5.1: Magnetic field for a single vortex calculated with our method.

accuracy of the truncated series, we use the degree of satisfaction of the Bogomolny equation as a measure of the error. Thus, we compute the magnetic field $B(x)$ for various values of ϵ and compare it with the right-hand side of equation (3.57c) $\frac{1}{2}(1-|\phi(x)|^2)$. The latter is computed using the truncated expansion of h and the parametrisation equation (4.24). More precisely we computed the L_2 and L_∞ norm of the difference:

$$L_2(N,\epsilon)=\left(\frac{1}{l_1l_2}\int_{\mathbb{T}^2}dx\,\left[B(x)-\frac{1}{2}(1-|\phi(x)|^2)\right]^2\right)^{1/2} \tag{5.23a}$$

$$L_\infty(N,\epsilon) = \max_x \left\{ \left| B(x) - \frac{1}{2}(1 - |\phi(x)|^2) \right| \right\} \tag{5.23b}$$

where N is the maximal order in the expansion. We have analysed the N and ϵ dependence of both quantities. Results are qualitatively the same for both, so we will choose L_2 to display. First, we will comment on the maximum precision, attained for $n = 51$. For $\epsilon \leq 0.6$ the L_2 norm is compatible with zero within machine precision (order $10^{-16,-17}$). Beyond this value L_2 becomes sizable and increases, reaching 10^{-4} at $\epsilon = 0.95$. For comparison we point out that with $N = 1$ the value (at $\epsilon = 0.95$) is $\mathcal{O}(10^{-2})$. Convergence is therefore slow in this case, but notice that the linear size of the box is 15 times the Debye screening length or 4.5 times the square root of the critical area.



(a) $L_2(N,\epsilon)$ versus N (in $\log_{10}$ scale) for $\epsilon = 0.9$.

(b) $a(\epsilon)$ versus ϵ compared to our best fit equation (3.5).

Figure 5.2: Large order behaviour of the series in the Bradlow parameter.

We performed a more systematic study by analysing the N dependence for fixed value of ϵ, in the range $0.1 - 0.95$. In this range the results are unaffected by the truncation in the number of Fourier modes $n_{\max}$. In all cases, we found that, beyond the first few orders, the dependence of $L_2(N,\epsilon)$ with N oscillates around an exponential fall-off. As an example, we show in the figure (5.2(a)) the $\epsilon = 0.9$ case . Therefore, we fitted $\log_{10} L_2$ data to the following linear function:

$$\log_{10} L_2(N,\epsilon) = a(\epsilon)N + b(\epsilon) \tag{5.24}$$

The parameter $a(\epsilon)$ is determined with errors reflecting the statistical and systematic uncertainties (range of fitting for example). Its value determines how the approximation improves when increasing the order N in the expansion. Its negative value is an indication that the expansion is indeed convergent. Obviously as ϵ increases so does $a(\epsilon)$. This is displayed in the figure (5.2(b)). For a convergent series and small ϵ one expects

$$\log_{10} L_2(N,\epsilon) \approx (N+1)\log_{10}(\epsilon) + \log_{10}(c_N) \tag{5.25}$$

and hence, $a(\epsilon) = \log_{10}(\epsilon) + \text{constant} + O(\epsilon)$. This is indeed the behaviour shown by the figure (5.1). Fitting $a(\epsilon)$ to a linear function of $\log_{10}\epsilon$ gives:

$$a(\epsilon) = 1.022(5)\log_{10}\epsilon - 0.0106(6) \tag{5.26}$$

Errors reflect the quality of the fit. Remarkably nothing seems to be happening at $\epsilon = 1$, where the area diverges. Data cannot be taken directly at $\epsilon = 1$ because they are severely affected by the truncation in the number of Fourier modes, but the behaviour up to $\epsilon = 0.95$ shows no sign of a change of pattern and extrapolates to $a(1) < 0$. Similar smooth behaviour is shown by $b(\epsilon)$. So we take our results as an indication that the series actually converges all the way up to $\epsilon = 1$.

Within our spirit of identifying the L_2 (or L_∞) norm of the equation with the error on the Higgs and magnetic field, we can use our data from a more practical viewpoint as an estimate of the number of terms required in the expansion to attain an a priori decided precision. An approximate formula can be derived from our data. If one is willing to compute the magnetic field with an error of 10^{-p} then the number of terms required in the expansion is given by:

$$n \approx \frac{1.83\epsilon + p - 3.25}{0.01 - 1.02\log_{10}\epsilon} \tag{5.27}$$

Although the formula gives a finite number even for $\epsilon = 1$, we stress once more that in practice at that very large volumes, truncation in the number of Fourier of terms would make the expansion increasingly computationally costly.

Now we will explore the implications of our expansion for large volumes. Our main assumption is that the solutions on the torus do converge to those on the plane. The convergence is expected to be fairly fast. A torus configuration is equivalent to a periodic array of vortices on the plane. However, vortices are exponentially localised objects so that if the period is large compared to the typical size of a vortex, the effect of the replicas is presumably very small. Now the convergence of the solution implies the following behaviour of the Fourier modes:

$$Y(\vec{p}) \equiv \frac{\hat{B}(p)}{4\pi q} = \lim_{\epsilon\to 1} \xi(1, n_1(p), n_2(p))\, h_{n_1(p),n_2(p)}\, (-1)^{n_1(p)+n_2(p)} \tag{5.28}$$

where $\hat{B}(\vec{p})$ is the Fourier transform of the magnetic field on the plane. The limit is taken at fixed p given by:

$$p^2 \equiv |\vec{p}|^2 = \xi\,(1-\epsilon) \tag{5.29}$$

where ξ is given by equation (5.10). This means that as ϵ tends to 1, the integers $n_i(p)$ have to grow. If instead, we take the limit $\epsilon \to 1$ keeping n_i fixed, the values should converge to $Y(0) = 0.5$ irrespective of n_i. In our case ($q = 1$), computing the value at $\epsilon = 1$ using our 51 orders and $n_1 = 1$,$n_2 = 0$ we get $Y(0) = 0.499947172199$. Worse results follow for higher modes (0.499572 for $n_1 = n_2 = 1$, 0.465354 for $n_1 = 2$, $n_2 = 1$, 0.387814 for $n_1 = n_2 = 2$, etc). The numerical agreement provides an additional hint that the expansion converges up to $\epsilon = 1$. It also indicates a poorer convergence for larger n_i (see later).

For p non-zero, equation (5.28) and the expectation of fast convergence, suggests that tuning ϵ and n_i in such a way that p is fixed we should obtain similar values. Only p, the modulus of $\vec{p}$, matters due to the cylindrical symmetry of the $q = 1$ solution on the plane. This is also satisfied by our expansion to a fairly high precision. For example, $Y(p)$ can be computed for $p^2 = \pi/20$ using $\epsilon = 0.975$ and $n_1 = n_2 = 1$, or $\epsilon = 0.95$ and $n_1 = 1, n_2 = 0$. From our expansion we get 0.39967 and 0.39977 respectively. This number is presumably very close ($< 1\%$) to $Y(p)$ on the plane. Similarly for $p^2 = \pi/10$ we get 0.326083 and 0.326075 from the same two modes and $\epsilon = 0.95, 0.9$ respectively. For $p^2 = \pi/5$ we get 0.22627 ($\epsilon = 0.8$), 0.22676 ($\epsilon = 0.9$) and 0.22608 ($\epsilon = 0.95$). In this way we can use our expansion to compute the Fourier transform of the magnetic field for a vortex on the plane with a precision of a few percent.

Now we will try to extract the consequences of the convergence of the expansion for the Fourier modes to a universal function of p as $\epsilon \to 1$. For very large n_i (large ξ) we can compute $Y(p)$ by taking $\epsilon = 1 - p^2/\xi$. Thus,

$$Y(p) \approx -\xi\,(-1)^{n_1+n_2}\,h_{n_1 n_2}(\epsilon) \approx -\xi \sum_{k=0}^{N} (-1)^{n_1+n_2}\,h^{(k)}_{n_1 n_2} e^{-p^2 \frac{k}{\xi}} \tag{5.30}$$

This suggests that for $\xi \to \infty$

$$h^{(k)}_{n_1 n_2} \longrightarrow \frac{-1}{\xi^2}\,\varphi(k/\xi)\,(-1)^{n_1+n_2} \tag{5.31}$$

where the function $\varphi(y)$ satisfies

$$Y(p) = \int_0^\infty dy\,\varphi(y)\,e^{-p^2 y} \tag{5.32}$$

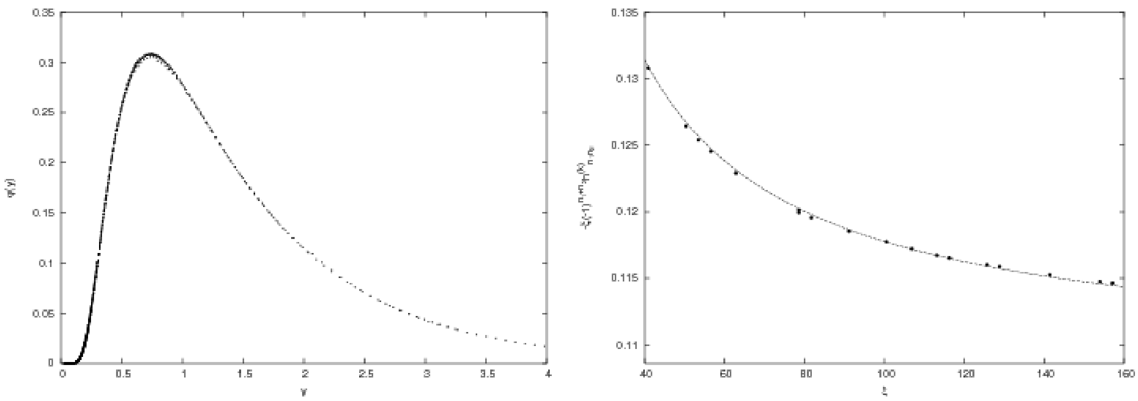
Fourier transforming we get:

$$B(x) = \int_0^\infty \frac{dy}{y} \varphi(y)\,e^{-|\vec{x}|^2/(4y)} \tag{5.33}$$

One can test these considerations by computing approximants to $\varphi(y)$ by using equation (5.31) for finite n_i. In the figure (5.3(a)) we show the shape obtained from the coefficients $h^{(k)}_{n_1 n_2}$. For any y we plot only those values of n_i such that $\xi \geq y/25$. The smoothness and small dispersion of values agrees with our expectations. We also investigated the way in which the limit is approached for large ξ. For example in the figure (5.3(b)) we plot $-\xi^2(-1)^{n_1+n_2} h^{(k)}_{n_1 n_2}$ for different values of n_i and a fixed value of $y = k/\xi = 1/\pi$. The solid line is the result of a fit to a function of the form $a + \frac{b}{\xi}$. Similar behaviour obtains for other y values. This analysis could be used to obtain a more precise estimation of the value $\varphi(y)$. For the time being we simply used the non-extrapolated shape shown in the figure (5.3(a)) and analysed the behaviour for large and small values of the argument y. For small y, the function is well described by $\exp\{a' - b'/y\}$ with a' and b' very close to 1. For large y the behaviour is also very well described by an exponential $\exp\{-a - by\}$. A fit in the range $y \in [2, 6]$ gives $a = 0.2973$ and $b = 0.9443$. Assuming our formula equation (5.33), we can, by saddle point methods, relate the large $|x|$ behaviour of $B(x)$ to these parameters. Indeed, b is predicted to be 1. The parameter a is given by $-\log(Z_1/2)$ where Z_1 was obtained numerically by de Vega and Schaposnik ($Z_1 = 1.7079$)[63], and recently Ref. [73] predicted its value to be $\log(2)/4$. These values of $Z_1/2$ differ by 10% from e^{-a}. This is a quite satisfactory agreement for the non-extrapolated curve obtained from the coefficients of our expansion. From formula 5.33 one can deduce a connection between our expansion and that of Ref. [63] in powers of $|x|^2$. The expression becomes

$$D_s = 2\frac{(-1)^s}{s!} \int_0^\infty dz\, z^{s-2}\,\varphi(\frac{1}{z}) \tag{5.34}$$

Numerically integrating the data we get $D_1 = -0.999976$, $D_2 = 0.747034$, $D_3 = -0.523573$, $D_4 = 0.36505$, in good agreement with Ref.[63] (-1, 0.72791, -0.48527, 0.31444 respectively).



(a) $\varphi(y)$ vs. y computed using equation (5.31). (b) $-\xi^2(-1)^{n_1+n_2}h^{(k)}_{n_1n_2}$ for different values of ξ and $y = k/\xi = 1/\pi$ fixed.

Figure 5.3: Scaling of the coefficients of the series in the Bradlow parameter.

From all the discussion above we see that our expansion, though originating from a small volume expansion on the torus, matches nicely with results known for the single vortex case at infinite area. Even though it might not give the same level of precision as other methods in that regime, it has the advantage of being readily generalisable to arbitrary fluxes and positions of the Higgs field zeroes. Furthermore, the same coefficients provide solutions in arbitrary torus sizes.

We can actually employ the previous formulas to estimate the error committed in $h_{n_1n_2}(\epsilon)$, the Fourier coefficients of the function h, as a result of the truncation of the series. The contribution of terms higher than N in the expansion, $\Delta_N h_{n_1n_2}(\epsilon)$, can be estimated in terms of the function $\varphi(k)$ and equation 5.31. The appropriate formula is

$$(-1)^{n_1+n_2}\,\Delta_{51}h_{n_1n_2}(\epsilon) = -\frac{1}{\xi}\int_{51/\xi}^{\infty} dy\,\varphi(y)\,\epsilon^{\xi y} \approx \frac{Z_1(e^{-1/\xi}\epsilon)^{51}}{2\xi(1-\xi\ln\epsilon)} \tag{5.35}$$

The last equality is obtained from the asymptotic behaviour of $\varphi(y)$ and, thus, is only valid for $(n_1^2 + n_2^2) < 10$. Applying this formula we get results which match with the discrepancies observed in some cases. For example, as we said before, $Y(0)$ $(\epsilon = 1)$ should be equal to 0.5 irrespective of which mode is used to compute it. However, our formula equation 5.35 predicts that the truncated evaluation up to 51 orders and $n_1 = n_2 = 2$ should fall short by 0.1105. The actual discrepancy found previously is 0.1122. Everything fits nicely. Our formulas can also be used to estimate the number of terms in the expansion required to obtain a given Fourier coefficient on the torus with a certain precision.

5.2.2 $q = 2$

Here we will apply our method to a multivortex situation. We take unit aspect ratio $(\tau = 1)$ and two units of flux. We can also use the procedure explained previously to fix the position of the zeroes of the Higgs field. We took the following points:

$$\left(0.35l_1, \frac{l_2}{2}\right); \left(0.65l_1, \frac{l_2}{2}\right); \tag{5.36}$$

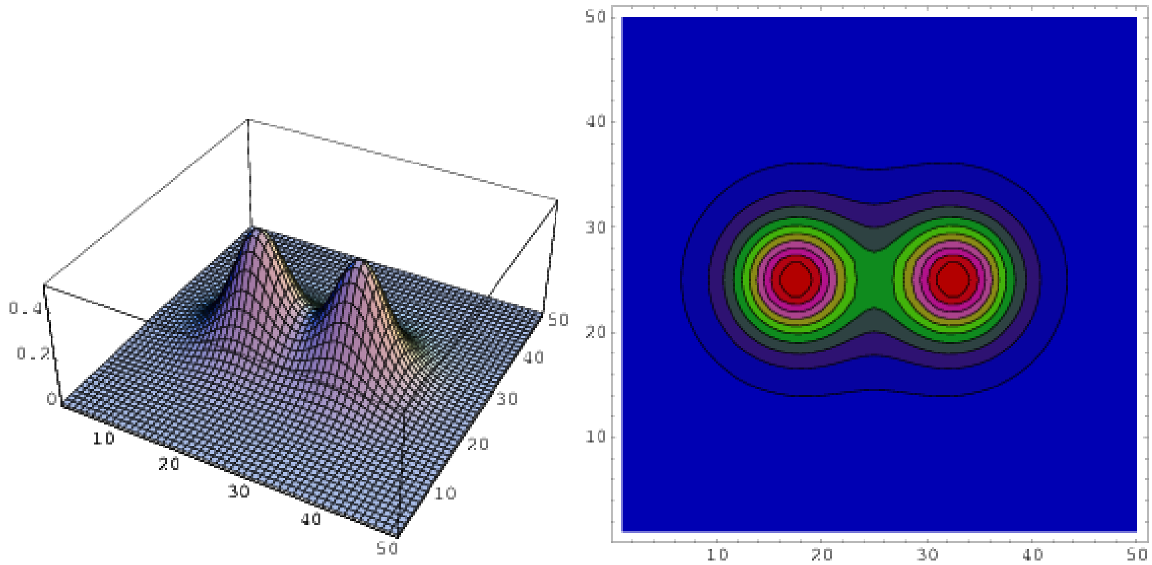
which are separated along the x direction a distance $0.3l_1$.

In the figures (5.4) we display the shape of the magnetic field obtained for $\epsilon = 0.9$ and 51 orders in the expansion. There is no particular difference in computational cost between this case and the unit flux one. The effects of truncation are similar to those obtained in the previous section: modes up to $\max(n_1, n_2) < 15$ are calculated up to machine precision. Note, however, that the position of the zeroes introduces a new scale in the problem, which translates into a typical scale for the modes. This might cause trouble if the zeroes are very close together.

We have repeated all our previous analysis of convergence with qualitatively identical results. For example, the L_2 norm of the function $(B(x) - \frac{1}{2}(1 - |\phi|^2))$, noted $L_2(N, \epsilon)$, seems to fall off exponentially fast with N, as in the $q = 1$ case. With similar definitions and methods to the ones used for $q = 1$ we got $a(\epsilon)$ and $b(\epsilon)$. Our best fit to the former quantity now gives:

$$a(\epsilon) = 1.032(6) \log_{10} \epsilon - 0.020(1) \tag{5.37}$$

Our previous conclusions about convergence extend to this case as well.



(a) $B(x)$ field for $\epsilon = 0.9$, and 51 orders. (b) Contour of $B(x)$ for $\epsilon = 0.9$, and 51 orders.

Figure 5.4: The magnetic field for a $q = 2$ configuration.

Focusing on the results for large areas, we emphasise that the main effect of a change in ϵ will the be to vary the separation between the vortices. Thus, with the coefficients obtained from our analysis we can actually explore the variations in shape for nearby vortices as a function of separation, a study which can have some interest (see for example Ref.[74]). In this $(\epsilon \sim 1)$ case, we can also compare with other alternative methods. In particular, in Ref. [75] the two vortex solution is computed by numerical methods. The finite square size used corresponds to $\epsilon = 0.91$ and the precision attained 10^{-4}. Our formulas give a precision in the $10^{-5} - 10^{-4}$ for this size, which is, at least, as good.

5.3 Solutions of the Extended Abelian Higgs Model

As we have seen in the section 4.4, the vortex equation for the extended abelian Higgs model in the torus is

$$\overline{\partial}\partial h = \frac{\epsilon}{2}\left(1 - e^{-2h} M_{ss'}\overline{\chi}_s\chi_{s'}\right) \tag{5.38}$$

where $M_{ss'}$ is an hermitian matrix, that can be written in terms of the homogeneous coordinates of the moduli space $c_s^{(a)}$

$$M_{ss'} = \sum_{a=1}^{N} \overline{c}_s^{(a)} c_{s'}^{(a)} \tag{5.39}$$

In principle one can follow exactly the same procedure used to solve the vortex equation in the non extended case, but with the difference that now

$$|\chi|^2 = M_{ss'}\overline{\chi}_s\chi_{s'} \tag{5.40}$$

All that we have to do is to replace the Fourier coefficients of $|\chi|^2$ with it's new value, given by

$$L(\vec{n}) = e^{-\frac{\xi(\vec{n})}{2}} e^{i\pi\frac{n_1 n_2}{q}} \left(\sum_s M_{s,s+n_1} e^{2\pi i \frac{s n_2}{q}}\right) \tag{5.41}$$

The rest of the numerical computation is exactly the same. It is important to note that the computational cost to solve with our Bradlow parameter expansion the solutions of the extended abelian Higgs model is the same as the cost required to solve the non extended solutions. This is so cause the coordinates of the moduli space enter in the vortex equation only through the $q \times q$ matrix $M_{ss'}$.

As we have seen in section 4.4.1, the moduli space for vortices living in the torus has a very different structure than vortices living in $\mathbb{R}^2$. This makes that the process to obtain the configurations in $\mathbb{R}^2$ more complicated than naively take the limit $\epsilon \to 1$. As we have seen, configurations with $q = 1$ in the torus are "trivial" in the sense that they reduce to the ANO vortices of the non extended model. We will then examine the case $N = 2$, and in detail the configurations with $q = 2$ in the torus. Later we will see how we can use these configurations to reproduce the Hindmarsh string-like solution in $\mathbb{R}^2$.

5.3.1 Configurations with $N = 2$ and $q = 2$

As we have seen in the section 4.4.2, the moduli space in this case is completely determined given three real parameters: the real number c, and the real and imaginary part of b. In terms of these numbers, the matrix $c_i^{(a)}$ is given by

$$c = \begin{pmatrix} 1 & b \\ 0 & c \end{pmatrix} \tag{5.42}$$

and the form of the hermitian matrix M that enters in the vortex equation

$$M = c^+ c = \begin{pmatrix} 1 & b \\ \overline{b} & c^2 \end{pmatrix} \tag{5.43}$$

we will systematically study some representative examples of this kind of configurations

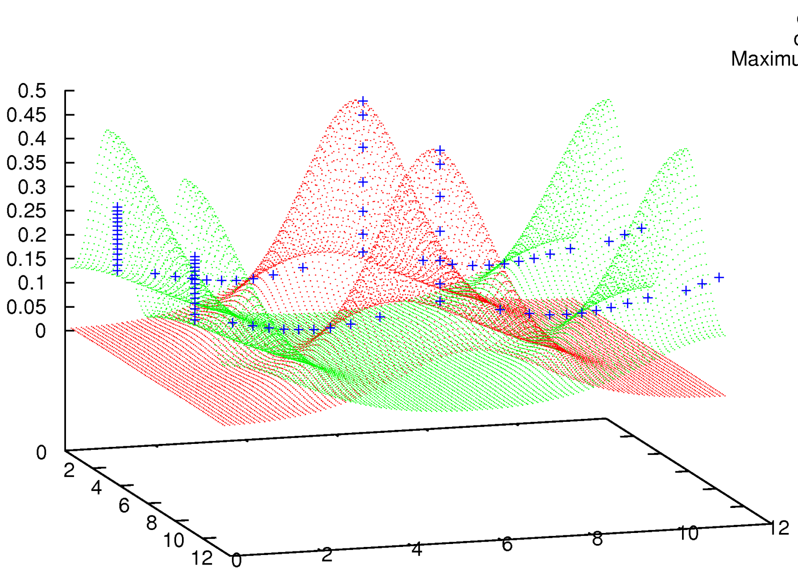


Figure 5.5: With red points the configuration with $c = 0$, in green the configuration with $c = \infty$, and with a blue $+$ sign the position and height of the maxima of the magnetic field along the trajectory from $c = 0$ to $c = \infty$.

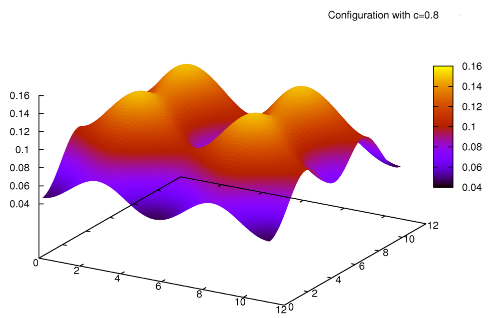
Diagonal form

In this section we will study a diagonal form for both the matrices c and M. In particular we will choose $b = 0$, and change the values of c.

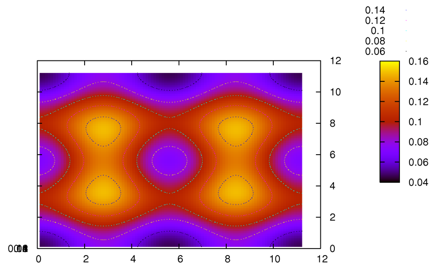
In the special case $c = 0$, we have a vortex solution of the non extended abelian Higgs model, with the zeros situated in the points $(l_1/4, l_2/2)$ and $(3l_1/4, l_2/2)$. The opposite situation, with $c = \infty$ is also an non extended abelian Higgs model configuration with the zeros located at $(l_1/4, 0)$ and $(3l_1/4, l_2)$. The intermediate values of c corresponds to different semilocal strings living in the torus. In the figure 5.5 we can see the form of the positions of the maxima of the magnetic field for intermediate values of c. We can see that at some point the two semilocal strings dissociate in four, in fact one can see one of such configurations in the figures 5.6.

This resembles the study of self dual configurations in the four dimensional torus, where the "individual" objects have a fractional topological charge. Here it can be argued that we are seeing 4 objects with a flux of $1/2$ each.

We can examine the maxima of the magnetic field as a function of the value of c. The first thing that we have to say is that there is a relation between configurations with some value of $c = k$ and configurations with $c = 1/k$. This is so because the configurations with $c = 0$ and $c = \infty$ are one a mere translation of the other. Since $c_i^{(a)}$ are homogeneous coordinates in the complex projective space we have the freedom to multiply them by an arbitrary complex



(a) Plot of the dissociated configuration.



(b) Contour plot of the dissociated configuration.

Figure 5.6: Dissociation of semilocal strings in the torus.

number. This means, in particular that the coordinates

$$c_i^{(a)} = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} \tag{5.44}$$

and

$$c_i^{(a)} = \begin{pmatrix} 1/c & 0 \\ 0 & 1 \end{pmatrix} \tag{5.45}$$

label the same point of the moduli space. But this last configuration is a mere translation of

$$c_i^{(a)} = \begin{pmatrix} 1 & 0 \\ 0 & 1/c \end{pmatrix} \tag{5.46}$$

In particular, the function that gives the value of the maximum of the magnetic field, $m(c)$ obeys

$$m(c) = m(1/c) \tag{5.47}$$

In the figure (5.7) we can see this data plotted. Points are plotted using each data point twice, one to compute the value of $m(c)$ and the other to compute $m(1/c)$. As we can see, the data perfectly fit in a unique function, as it should. For $c \lesssim 0.65$ we see two vortices located at $(l_1/4, l_2/2)$ and $(3l_1/4, l_2/2)$. For $0.65 \lesssim c \lesssim 1.55$ there are four structures, this is the region where the four dissociated objects appear. For $c \gtrsim 1.55$ there are again two vortices located at $(l_1/4, 0)$ and $(3l_1/4, l_2)$. As expected, the points where the dissociation takes place are one the inverse of the other.

A giant vortex case

In this section we will again vary the parameter c of the matrix M, but with a different value of the parameter b. This parameter will be chosen so that the first component of the Higgs field has a double zero at the centre of the torus. This special value correspond to $b = \sqrt{2} - 1$, so the form of the matrix is given by

$$c = \begin{pmatrix} 1 & \sqrt{2} - 1 \\ 0 & c \end{pmatrix} \tag{5.48}$$

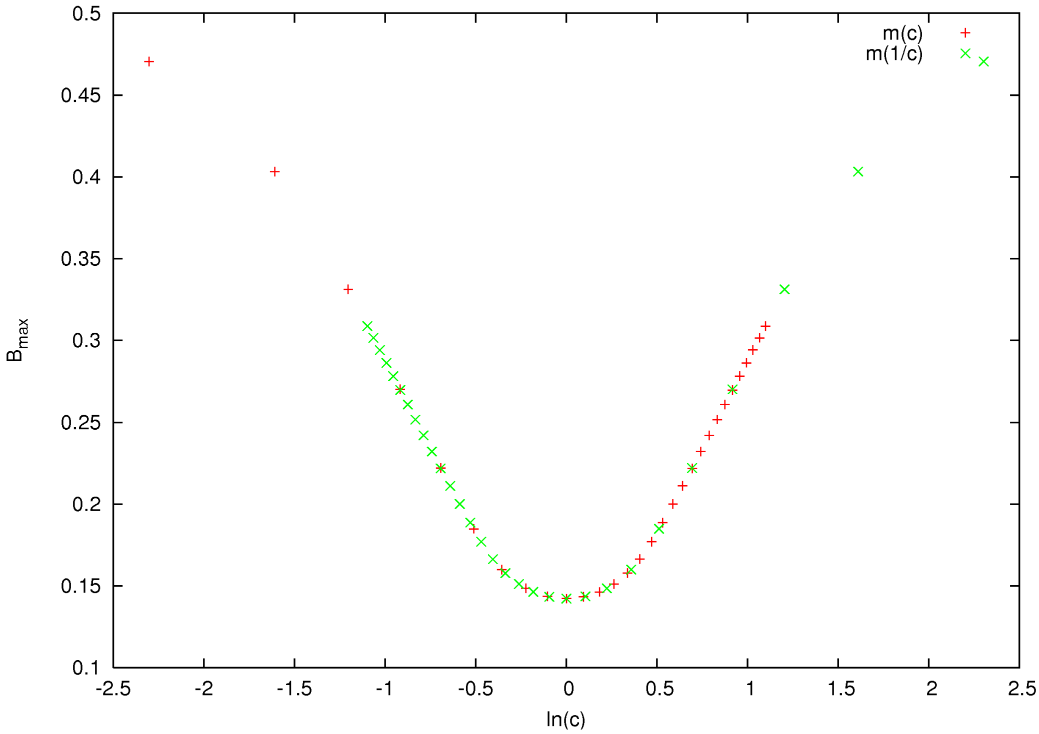


Figure 5.7: Height of the magnetic field at its maximum as a functions of $\ln c$. We can see the symmetry under the change $c \to 1/c$.

we will vary the value of the parameter c, starting with $c = 0$, where the solution corresponds to a non extended double flux vortex located at the centre of the torus, to $c = \infty$, where the solution, as we have said in the previous section corresponds to a $q = 2$ non extended vortex located at $(l_1/4, 0)$ and $(3l_1/4, l_2)$.

As we can see in the figure (5.8), we have just a single maximum when $c = 0$, but as soon as c has a value different from zero the configuration starts to have two maxima.

Later, when $c \sim 0.85$, the two vortex-like structures breaks again in four, having again the same dissociation phenomena as in the previous case. For $0.85 \lesssim c \lesssim 1.55$ we have four semilocal strings. Finally for $c \gtrsim 1.55$ we have again two maxima located at $(l_1/4, 0)$ and $(3l_1/4, l_2)$.

This case, again, shows us that the interpretation of the moduli relies in four structures that can get combined in two or even a single object.

5.3.2 Obtaining configurations for $\mathbb{R}^2$

In this section we will try to reproduce the solution found by Hindmarsh in [2] (see also [1, 70]), that we have described in the section 3.6.4, of a semilocal string with variable size living in $\mathbb{R}^2$. The most important feature of this solution is that it has a cylindrical symmetry, and the Higgs field is given by

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \frac{1}{\sqrt{r^2 + |q_0|^2}} \begin{pmatrix} re^{\imath\theta} \\ q_0 \end{pmatrix} \exp\left\{ \frac{1}{2} u(r; |q_0|) \right\} \tag{5.49}$$

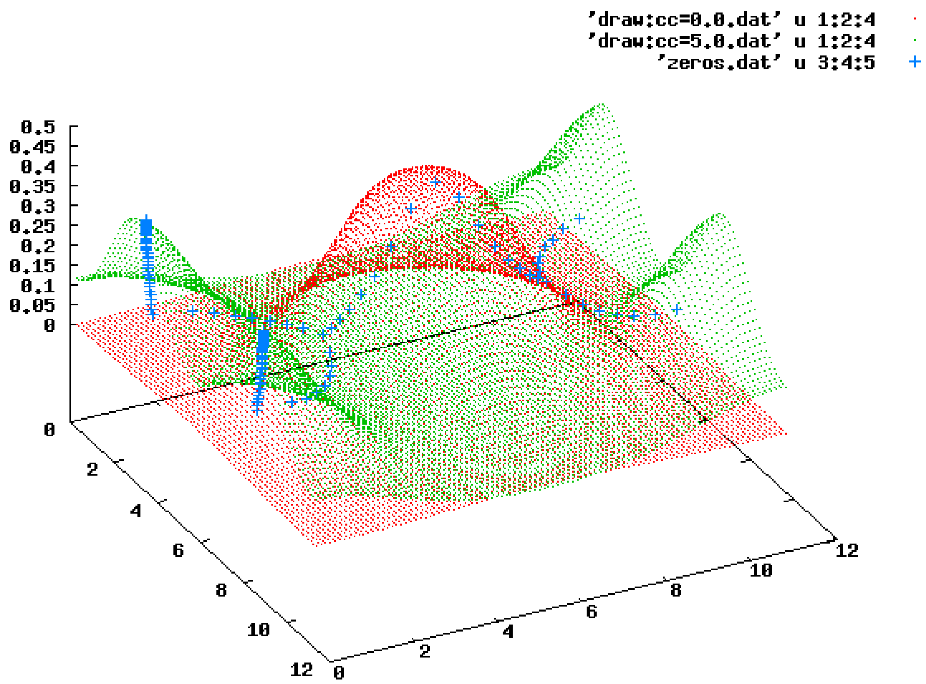


Figure 5.8: The giant vortex case. With red points the configuration with $c = 0$, in green the configuration with $c = \infty$, and with a blue + sign the position and height of the maxima of the magnetic field along the trajectory from $c = 0$ to $c = \infty$.

where one needs to solve a differential equation to obtain $u(r)$ (more details in 3.6.4). The complex parameter q_0 can be interpreted as the "width" of the vortex, in the sense that the magnetic field decays asymptotically as

$$B \sim \frac{|q_0|^2}{r^4} + \dots \tag{5.50}$$

the phase of q_0 gives the orientation of the semi local string at the origin. It is important to note here that the first component of the Higgs field has a zero in the origin, and the second component has the zero at infinity.

To reproduce this solution in the Torus, we will use a $q = 2$ and $\tau = 2$ torus, with the zeros of the first component of the Higgs field in the point $Z_c = \frac{l_1}{4}(1 + \imath\tau/2)$ and $\frac{l_1}{4}(1 + 3\imath\tau/2)$, and the zeros of the second component of the Higgs field at $Z_p = \imath\frac{l_1}{8}\tau$ and $3\imath\frac{l_1}{8}\tau$ (see figure (5.9)). This case corresponds with the case of a diagonal form of both M and c.

Clearly in the limit $\epsilon \to 1$, and focusing in the vicinity of one of the zeroes of ϕ_1, all the other zeros of the Higgs field approach infinity, just as desired. Because of the symmetry in the position of the zeros (the distance from one zero to its neighbourhoods is the same along the X and Y axis), the final configuration must be symmetric under the interchange of the X

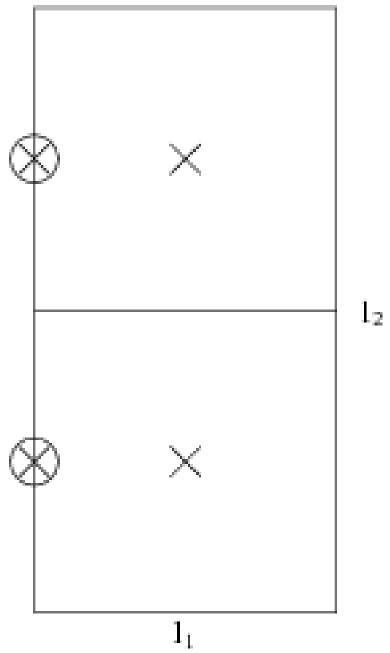


Figure 5.9: Zeros of the first component of the Higgs field (marked with a $\times$), and of the second Higgs component (marked with a $\otimes$)

axis and the Y axis. For that purpose we will write

$$\phi_1 = \sqrt{\epsilon} e^{-H} (\cos\alpha) \chi_1 \tag{5.51a}$$
$$\phi_2 = \sqrt{\epsilon} e^{-H} (\sin\alpha) \chi_2 \tag{5.51b}$$

To have the zeros in the correct position ($z = Z_p + re^{\imath\theta}$ where $Z_T = \frac{l_1}{4}(1 + \imath\tau)$)

$$\chi_1(z) = Ne^{fz(z-\bar{z})} \vartheta_3 \left(\frac{2\pi(z + Z_T - Z_c)}{l_1} | \imath\tau \right) \vartheta_3 \left(\frac{2\pi(z + Z_T + Z_c)}{l_1} | \imath\tau \right) \tag{5.52a}$$
$$\chi_2(z) = Ne^{fz(z-\bar{z})} \vartheta_3 \left(\frac{2\pi(z + Z_T - Z_p)}{l_1} | \imath\tau \right) \vartheta_3 \left(\frac{2\pi(z + Z_T + Z_p)}{l_1} | \imath\tau \right) \tag{5.52b}$$

where N is a normalisation constant, that should be equal for the two functions χ_1 and χ_2 because one is a mere translation of the other. From the Hindmarsh solution, and computing $|\phi_2/\phi_1|$ we have

$$\frac{|q_0|}{r} = \left| \frac{\chi_2(Z_p)}{\chi_1(Z_p)} \right| (\tan\alpha) \tag{5.53}$$

close to $r = 0$, we have

$$\chi_2(z) = Ne^{\pi\tau/4} \vartheta_3(0|\imath\tau) \vartheta_2(0|\imath\tau) + \mathcal{O}(z - Z_p) \tag{5.54a}$$
$$\chi_1(z) = \imath Ne^{\pi\tau/4} \vartheta_2(0|\imath\tau) \vartheta_3(0|\imath\tau) \vartheta_4^2(0|\imath\tau) \frac{2\pi}{l_1} (z - Z_p) + \mathcal{O}(z - Z_p)^2 \tag{5.54b}$$

so the relation between q_0 and α is

$$|q_0| = \frac{1}{\sqrt{\pi}\, |\vartheta_4(0|2\imath)|^2} \frac{\tan\alpha}{\sqrt{1 - \epsilon}} \tag{5.55}$$

The numerical value of the pre-factor is given by

$$F = \frac{1}{\sqrt{\pi}\,|\vartheta_4(0|2\imath)|^2} = 5.68\ 42\ 76\ 78\ 86\ 20\ 94\ 67 \times 10^{-1} \tag{5.56}$$

Inverting the relation, we obtain

$$\epsilon = 1 - \frac{F^2}{|q_0|^2}\tan^2\alpha \tag{5.57}$$

Combining different values of α and ϵ, we should see, in the limit $\epsilon \to 1$, the same physical object living in $\mathbb{R}^2$: The Hindmarsh solutions. To see that this is, in fact, the case, we will choose a fixed value of q_0, and we will plot the value of the magnetic field for different values of both α and ϵ, in such way that the value of the physical parameter q_0 is constant. We will choose the value $q_0 = 0.5$.

So to proceed we will do the following: We choose a value of r (the distance to the centre of the object), and we plot the value of the magnetic field at distance r from the centre along two directions (one the X axis, and the other along the line $Y = X$), with an error measuring the efect of the truncation of the ϵ series (this error is computed as $|B - \frac{1}{2}(1 - |\phi_1|^2 - |\phi_2|^2)|$). We do this for different pair of values (ϵ, α). Next we have to extrapolate the values up to $\epsilon = 1$. To do this we will assume that the function that we are trying to predict is analytic, and use a power expansion of the form

$$B(r,\epsilon) = \sum_n a_n(r)(1-\epsilon)^n \tag{5.58}$$

We use as many terms in this expansion as we need to obtain a good χ^2 per degree of freedom (in practice no more than 5 terms). This gives us a value for the magnetic field for the Hindmarsh configuration. We can make the extrapolation with different number of terms in the series, and the differences will reflect the systematic error due to the following facts

1. We do not know the value of the magnetic field $B(r,\epsilon)$, due to the fact that we truncate the ϵ expansion.

2. We do not know the form of the function $B(r,\epsilon)$.

An example of this way of proceed can be seen in the figure (5.10)

The error bars measuring the effect of the truncation in the ϵ series is bigger when r is small, and then the points with small value of r are computed with more error. To alleviate this effect, we will study in detail the form of the ϵ series for small values of r. If we fix the value of α, the series has the form

$$B(\alpha,r,\epsilon) = \sum_{n=1}^{\infty} B^{(n)}(\alpha,r)\epsilon^n \tag{5.59}$$

A typical case of the form of the contributions to the magnetic field can be seen in the figure (5.11).

We have computed the first 51 orders in the ϵ expansion, and the large orders $n > 15$ have a very nice fitting with a line (in log scale). We will compute the contribution of the rest of the series (from order 51 to ∞) as if this behaviour of the coefficients were true for the rest of the series. That is to say, we will assume that for $n > 51$ the coefficients have the form

$$\log\left(-B^{(n)}(\alpha,r)\right) = an + b \tag{5.60}$$

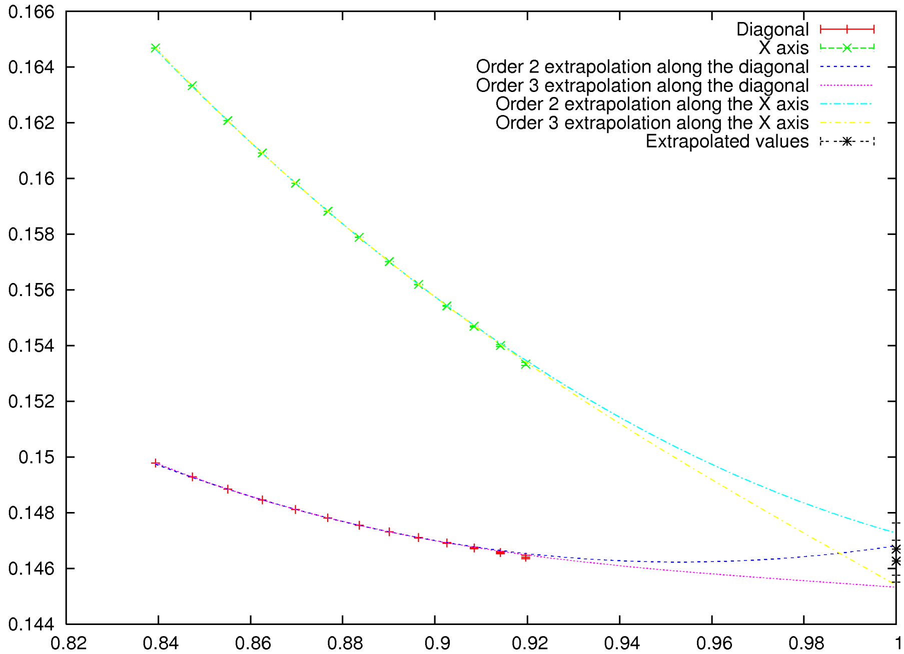


Figure 5.10: Extrapolating $B(r, \epsilon)$ up to $\epsilon = 1$ along the diagonal ($+$ points), and the X axis ($\times$ points). Here we use the special value $r = 2.0$

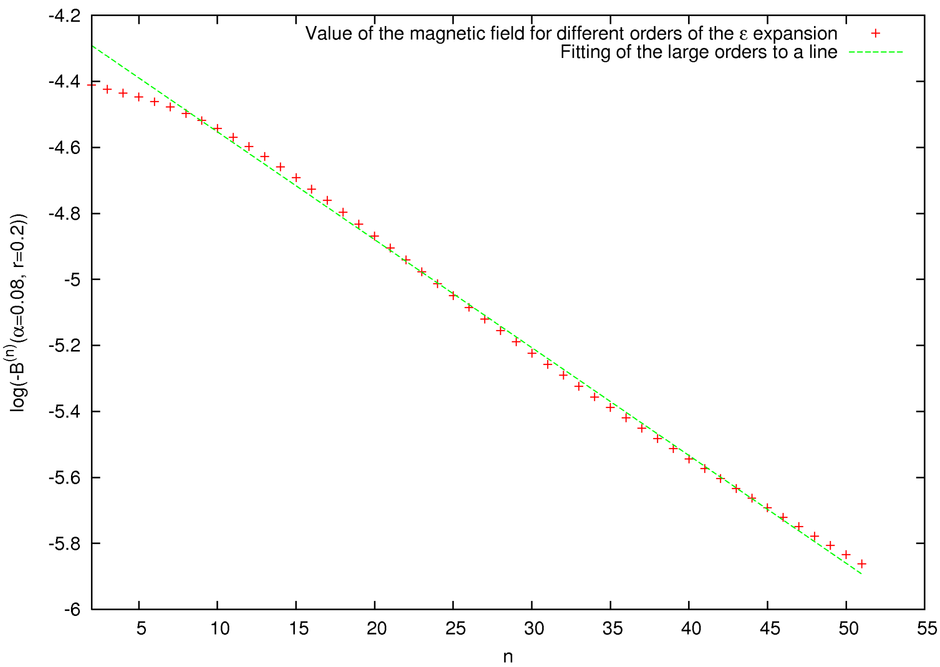


Figure 5.11: The contribution to the magnetic field to order N in the ϵ expansion ($\log$ scale) versus N. The line represent a fitting for the large orders. In this example the values $\alpha = 0.08$ and $r = 0.2$ has been used.

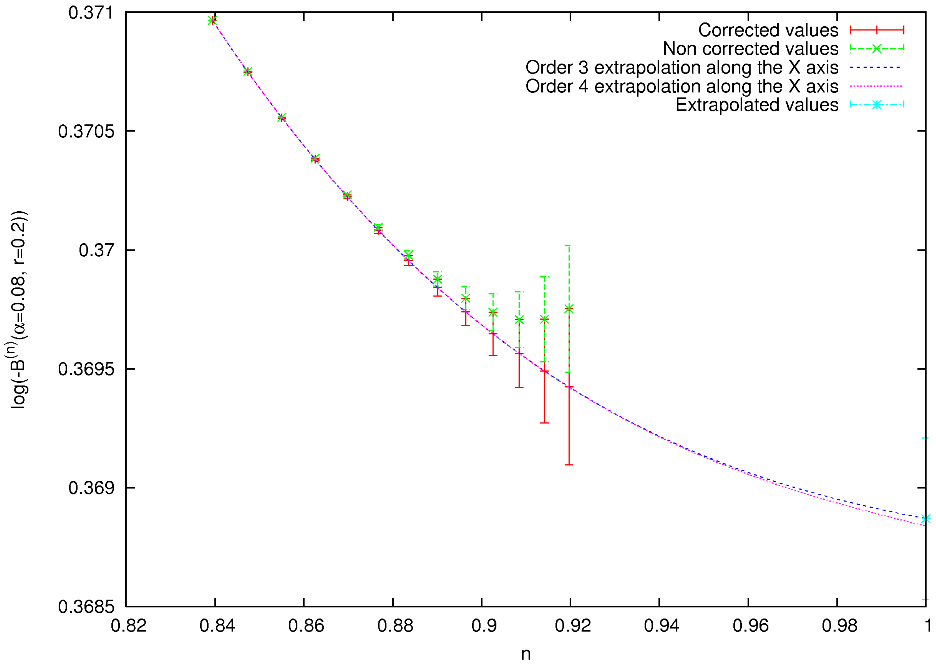


Figure 5.12: In this figure we can see the effect of the correction of equation (5.61). The non corrected values are marked with a $\times$, and the corrected values with a $+$. The extrapolation of $B(r, \epsilon)$ up to $\epsilon = 1$ is done using the corrected points. This figure shows the result for $r = 0.2$.

in this case the sum of all the orders from 51 to infinity is given by

$$\Delta B(\alpha, r) = -e^b \frac{(\epsilon e^a)^{52}}{1 - \epsilon e^a} \tag{5.61}$$

Thus we can correct the position of the points using the formulae

$$B(\alpha, r, \epsilon) = \sum_{n=1}^{51} B^{(n)}(\alpha, r)\epsilon^n + \Delta B(\alpha, r) \tag{5.62}$$

Of course we do not expect that the correction given by equation (5.61) is exact, but if the form of the coefficients of the series does not change abruptly just very near of the order 51, and they still continue to decrease, although in a different way for very large orders, the estimation given in equation (5.61) is appropriate. To measure the error made by this correction, we will weigh the error of the corrected point with a value equal to the correction. We will use the corrected points as our source to make the extrapolation of the magnetic field up to $\epsilon = 1$. In the figure (5.12) we can see the effect of the correction, and the value of the extrapolation using the corrected points. It is very important to note that the χ^2 per degree of freedom decreases when the correction to the points is applied, and that the error of the extrapolated data also decreases. We have corrected in this way all the points that have a big error, namely, those with r small.

At the end of the day, the extrapolated values of the data along the X axis, and along the line $Y = X$ should be equal within errors, obtaining an $\mathbb{R}^2$ configuration with spherical symmetry. This is, indeed, the case, as we can see in the plot of the magnetic field computed

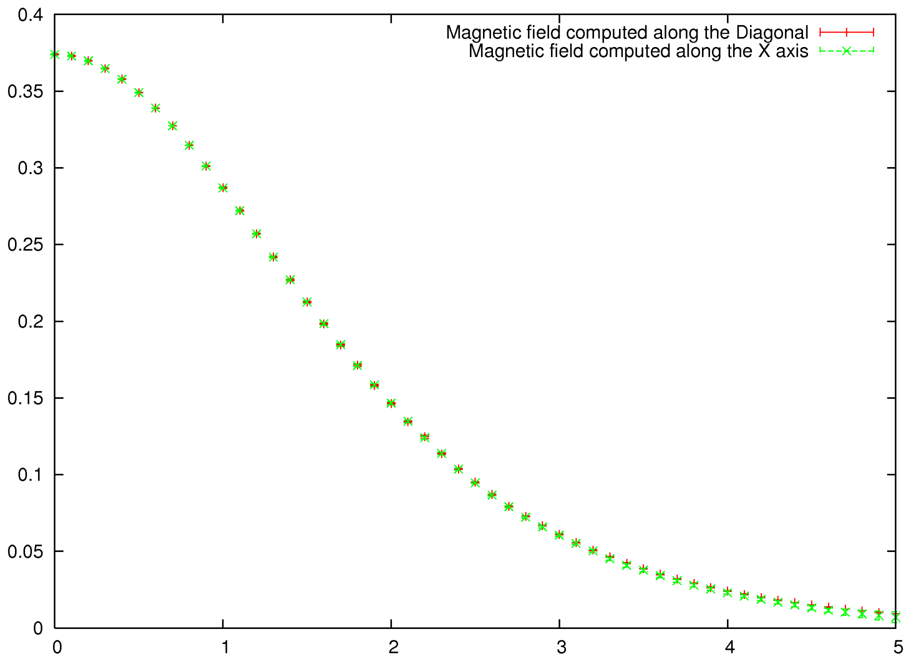


Figure 5.13: Plot of the magnetic field computed with the extrapolated data.

in this way (Figure (5.13)). The size of the errors are at most of the order of a 0.2% in the final extrapolated data for both the data computed along the X axis and along the line $Y = X$. We can thus, conclude that we have reproduced again the results of $\mathbb{R}^2$ with our data obtained from the torus.

5.3.3 Other way of computing the Hindmarsh solution

In this section we will explore other way of computing the cylindrically symmetric solution described in the section 3.6.4. As we have seen the differences in the moduli space structure for vortices living in the torus and vortices living in $\mathbb{R}^2$ makes impossible to obtain a one parameter family of solutions with arbitrary size for the case of $q = 1$ vortices living in the torus. To mimic the $\mathbb{R}^2$ configurations in the previous section we have used the trick of computing a $q = 2$ solution, and look only to the region very close to one of the zeros of one Higgs component. In the limit $\epsilon \to 1$, the other components of the Higgs fields have their zeros at infinity, and their contribution in the interesting region is a mere constant.

Since we are really interested only in the limit $\epsilon \to 1$, and in this limit

$$\chi_2(z) = Ne^{\pi\tau/4}\vartheta_3(0|\imath\tau)\vartheta_2(0|\imath\tau) + \mathcal{O}(\sqrt{1-\epsilon}) \tag{5.63a}$$

$$\chi_1(z) = \imath Ne^{\pi\tau/4}\vartheta_2(0|\imath\tau)\vartheta_3(0|\imath\tau)\vartheta_4^2(0|\imath\tau)\frac{2\pi}{l_1}(z - Z_p) + \mathcal{O}(1-\epsilon) \tag{5.63b}$$

close to Z_p that is a zero of χ_1, we can replace χ_2 by a constant value. We will now use a $q = 1$ configuration, but with a *modified vortex equation*

$$\partial\overline{\partial}h = \frac{\epsilon}{2}\left[1 - e^{-2h}\left(|\chi_1|^2 + |\tilde{q}_0|^2\right)\right] \tag{5.64}$$

the complex constant $\tilde{q}_0$ is replaced by the value of $\sin\alpha\chi_2(Z_p)$, so we have the relation

$$|\tilde{q}_0|^2 = N^2 \sin^2 \alpha \, e^{\pi\tau/2} |\vartheta_3(0|\imath\tau)|^2 |\vartheta_2(0|\imath\tau)|^2 \tag{5.65}$$

The relation between the physical parameter q_0 of the $\mathbb{R}^2$ configuration and our parameter $\tilde{q}_0$ is given by

$$\frac{|\tilde{q}_0|^2 F^2}{N^2 e^{\pi\tau/2} |\vartheta_3(0|\imath\tau)|^2 |\vartheta_2(0|\imath\tau)|^2 - 1} = |q_0|^2 (1-\epsilon) \tag{5.66}$$

where F is given in equation (5.56).

Again we can compute the $q = 1$ solution in our torus for different values of the parameter $\tilde{q}_0$, and then plot the value of the magnetic field (or of any quantity that we want to compute) for different values of ϵ and $\tilde{q}_0$, so that the physical parameter q_0 remains constant. Again we can extrapolate this data up to $\epsilon = 1$, and obtain in this way the desired quantities of the $q = 1$ configurations in $\mathbb{R}^2$.

We have done this work, and the computed values of the magnetic field, agrees within errors with the values computed in section 5.3.2. This alternative method based upon substituting the component of the Higgs field that we send to infinity by a constant near the region in which we are interested can be useful to mimic $\mathbb{R}^2$ configurations easily.

Six

Vortex dynamics and the metric on the moduli space.

For static vortices, the energy does not depend on the position of the zeros of the Higgs field, but if we consider moving vortices, the kinetic energy will depend on both the position and the velocity of the zeros of the Higgs field. This fact destroy the naive interpretation of critically coupled vortices as non interacting objects. This kinetic energy, induces a metric in the moduli space, that is able to describe the dynamics (collisions) of vortices in some limit.

In this chapter we will show how the kinetic energy induces a metric in the moduli space, its most important properties, and how to compute this metric.

6.1 The metric of the moduli space

The energy of a field configuration (A_i, ϕ) can be divided into potential and kinetic energy

$$E[A_i, \phi] = T[A_i, \phi] + V[A_i, \phi] \tag{6.1}$$

If we choose some multivortex solution of the Bogomolny equations $(A_i^{(v)}, \phi^{(v)})$, parametrised by some arbitrary complex coordinates $\{\zeta_i, \overline{\zeta}_i\}$ of the moduli space, like, for example the position of the zeros of the Higgs field, or the coordinates $\{c_i, v\}$ if we are in the Torus $\mathbb{T}^2$, we know that the potential energy is a constant given by the Bogomolmy bound:

$$V[A_i^{(v)}, \phi^{(v)}] = q\pi \tag{6.2}$$

and that the kinetic energy, that in the $A_0 = 0$ gauge is given by:

$$T[A_i^{(v)}, \phi^{(v)}] = \frac{1}{2}\int_{\mathbb{T}^2} d^2x \, \left[\dot{A}_i^2 + |\dot{\phi}|^2\right] \tag{6.3}$$

is, in general, a quadratic form in the coordinates $\dot{\zeta}_i$

$$T[A_i^{(v)}, \phi^{(v)}] = \frac{\pi}{2} g_{ij} \, \dot{\zeta}_i \dot{\overline{\zeta}}_j \tag{6.4}$$

so, in this context, the kinetic energy induces in a natural way a metric in the moduli space ($\mathbb{C}^q$ in $\mathbb{R}^2$, $\mathbb{CP}^{q-1}$ if we are in the Torus). We are interested in the general properties of this metric, and in how to compute it. We will also show that this metric gives us the dynamics of colliding vortices in the non relativistic approximation.

6.2 The Samols method for computing the metric

As we have said before, the kinetic energy is given by

$$T[A_i^{(v)}, \phi^{(v)}] = \frac{1}{2}\int_{\mathbb{R}^2} d^2x \, \left[\dot{A}_i^2 + |\dot{\phi}|^2\right] \tag{6.5}$$

and the equations that the functions $(\dot{A}_i, \dot{\phi})$ have to obey are given by the time derivative of the Bogomolny equations. As we have done in section 3.5, we will parametrise the Higgs and gauge field

$$\begin{aligned} \phi &= e^H \\ A &= \imath\partial\overline{H} \end{aligned} \tag{6.6}$$

where $H = \frac{1}{2}h + \imath g$. Now the first Bogomolny equation that reads

$$(\overline{\partial} - \imath\overline{A})\phi = 0 \tag{6.7}$$

is automatically solved by our parametrisation, and its time derivative can be written as

$$\dot{\overline{A}} = -\imath\overline{\partial}\dot{H} \tag{6.8}$$

the time derivative of the Higgs field can be written

$$\dot{\phi} = \phi\dot{H} \tag{6.9}$$

so the expression for the kinetic energy is

$$T = \frac{1}{2}\int_{\mathbb{R}^2} d^2x \, \left[\partial\dot{\overline{H}}\,\overline{\partial}\dot{H} + e^h|\dot{H}|^2\right] \tag{6.10}$$

The function h can be obtained solving the vortex equation

$$-\partial\overline{\partial}h + 1 - e^h = 4\pi\sum_{r=1}^{q}\delta^2(x - x_r) \tag{6.11}$$

and the function $\dot{H}$ can be obtained from two different sources. The first is the Gauss Law

$$\partial_i F^{0i} = \frac{\imath}{2}\left(\phi\partial_0\overline{\phi} - \overline{\phi}\partial_0\phi\right) \tag{6.12}$$

that in our notation can be written as

$$\left(-\partial\overline{\partial} - e^h\right)\dot{g} = 0 \tag{6.13}$$

the other is the time derivative of the vortex equation. Away from any zero of the Higgs field

$$\left(-\partial\overline{\partial} - e^h\right)\dot{h} = 0 \tag{6.14}$$

equations (6.13) and (6.14) can be written together as

$$\left(-\partial\overline{\partial} - e^h\right)\dot{H} = 0 \tag{6.15}$$

Again this equation is only valid far away from the zeros of the Higgs field. When we are close to the zero Z_r the function H have the behaviour $H \approx \log(z - Z_r)$, and taking the time derivative we have

$$\dot{H} \approx \frac{-\dot{Z}_r}{z - Z_r} \tag{6.16}$$

Now, recall that $-\partial\overline{\partial} \log|z - Z_r|^2 = 4\pi\delta^2(z - Z_r)$. Then

$$-\partial\overline{\partial}\left(\frac{-1}{z - Z_r}\right) = 4\pi\frac{\partial}{\partial Z_r}\delta^2(z - Z_r) = -4\pi\partial\delta^2(z - Z_r) \tag{6.17}$$

now taking into account that $e^h\dot{H}$ has no singularities, because the zero of the Higgs field cancels the pole of $\dot{H}$, we can write the complete version of equation (6.15)

$$\left(-\partial\overline{\partial} - e^h\right)\dot{H} = -4\pi\sum_{r=1}^{q}\dot{Z}_r\partial\delta^2(z - Z_r) \tag{6.18}$$

Now differentiating the vortex equation with respect to the position of the Higgs zero Z_s, we obtain

$$\left(-\partial\overline{\partial} - e^h\right)\frac{\partial h}{\partial Z_s} = -4\pi\partial\delta^2(z - Z_s) \tag{6.19}$$

comparing this last equation with (6.18), and noting that the operator $\left(-\partial\overline{\partial} - e^h\right)$ is invertible (has no zero modes), we conclude

$$\dot{H} = \sum_{r=1}^{q}\frac{\partial h}{\partial Z_r}\dot{Z}_r \tag{6.20}$$

Now we come back to our formula for the energy (6.10), and we will split the integral in two parts: the first, that we will call D is a union of discs D_r, of radius ε centred at the zeros of the Higgs field Z_r. It is clear that choosing ε small enough we can always make all the discs D_r non overlapping. The second part is the contribution for the rest of the space $\mathbb{R}^2 - D$, where the functions H and $\dot{H}$ have no poles.

$$T = \frac{1}{2}\int_{\mathbb{R}^2 - D} d^2x\ \left[\partial\overline{\dot{H}}\,\overline{\partial}\dot{H} + e^h|\dot{H}|^2\right] + \frac{1}{2}\int_D d^2x\ \left[\partial\overline{\dot{H}}\,\overline{\partial}\dot{H} + e^h|\dot{H}|^2\right] \tag{6.21}$$

The second integral is zero in the limit $\varepsilon \to 0$. The first integral can be integrated by parts obtaining

$$T = \frac{1}{2}\int_{\mathbb{R}^2 - D} d^2x\ \partial\left(\overline{\dot{H}}\,\overline{\partial}\dot{H}\right) + \frac{1}{2}\int_{\mathbb{R}^2 - D} d^2x\ \overline{\dot{H}}\left[-\partial\overline{\partial} + e^h\right]\dot{H} \tag{6.22}$$

and since we have subtracted the region of the zeros of the Higgs field, the second integral is zero by equation (6.15). We can apply Stokes theorem to the first integral and obtain

$$T = -\frac{\imath}{4}\sum_r\int_{C_r} d\overline{z}\,\overline{\dot{H}}\,\overline{\partial}\dot{H} \tag{6.23}$$

where C_r is the boundary of the disc D_r. This means that we only need to know $\dot{H}$ near the zeros of the Higgs field, so we are going to make an expansion of h near a zero of the Higgs field

$$\begin{aligned} h &= \log|z - Z_s|^2 + a_s + \frac{\overline{b}_s}{2}(z - Z_s) + \frac{b_s}{2}(\overline{z} - \overline{Z}_s) + \\ &+ \overline{c}_s(z - Z_s)^2 + c_s(\overline{z} - \overline{Z}_s) + d_s|z - Z_s|^2 + \mathcal{O}(|z - Z_s|^3) \end{aligned} \tag{6.24}$$

Substituting this expression in the vortex equation (6.11) we can easily see that $d_s = -\frac{1}{4}$. The other coefficients are determined by the positions of the other zeros of the Higgs field Z_r. Differentiating we obtain

$$\begin{aligned}\frac{\partial h}{\partial Z_r} &= -\frac{\delta_{rs}}{a - Z_s} + \frac{\partial a_s}{\partial Z_r} - \frac{1}{2}\bar{b}_s\delta_{rs} + \frac{1}{2}\frac{\partial \bar{b}_s}{\partial Z_r}(z - Z_r) + \frac{1}{2}\frac{\partial b_s}{\partial Z_r}(\bar{z} - \overline{Z}_r) \\ &- 2c_s(z - Z_s)\delta_{rs} + \frac{1}{4}(\bar{z} - \overline{Z}_s)\delta_{rs}\mathcal{O}(z - Z_s) \end{aligned} \tag{6.25}$$

so we have

$$\dot{\overline{H}} = \frac{-\dot{\overline{Z}}_s}{\bar{z} - \overline{Z}_s} + \mathcal{O}(1) \tag{6.26}$$

and

$$\bar{\partial}\dot{H} = \sum_{r=1}^{q} \dot{Z}_r \left(\frac{1}{2}\frac{\partial b_s}{\partial Z_r} + \frac{1}{4}\delta_{rs}\right) \tag{6.27}$$

and using the residue theorem we can write our final expression for the kinetic energy

$$T = \frac{\pi}{8}\sum_{r,s=1}^{q}\left(\delta_{rs} + 2\frac{\partial b_s}{\partial Z_r}\right)\dot{Z}_r\dot{\overline{Z}}_s \tag{6.28}$$

so the metric is given by

$$ds^2 = \frac{1}{4}\sum_{r,s=1}^{q}\left(\delta_{rs} + 2\frac{\partial b_s}{\partial Z_r}\right)dZ_r d\overline{Z}_s \tag{6.29}$$

6.3 General properties of the metric

Since the kinetic energy is a real quantity, the metric is hermitian

$$g_{rs} = \bar{g}_{sr} \tag{6.30}$$

this means that the coefficients b_r have the symmetry property

$$\frac{\partial b_s}{\partial Z_r} = \frac{\partial \bar{b}_r}{\partial \overline{Z}_s} \tag{6.31}$$

and this result can be used to prove that the metric is Kähler, a property first noted by Ruback (See appendix B of [75]). A metric is Kähler iif the associated two form

$$\hat{w} = \frac{\imath}{2}g_{rs}\, dZ_r \wedge d\overline{Z}_s \tag{6.32}$$

is closed. For the components of the metric this means

$$\frac{\partial g_{rs}}{\partial Z_l} = \frac{\partial g_{ls}}{\partial Z_r} \tag{6.33a}$$

$$\frac{\partial g_{rs}}{\partial \overline{Z}_l} = \frac{\partial g_{rl}}{\partial \overline{Z}_s} \tag{6.33b}$$

a property that can be checked straightforwardly using (6.31). The Kähler property can be used to write the metric using a centre of mass coordinate and relative coordinates. If we define

$$\begin{aligned} Z &= \frac{1}{q}\sum_r Z_r && (6.34a) \\ w_r &= Z_r - Z && (6.34b) \end{aligned}$$

using the Kähler property the metric takes the form

$$ds^2 = \frac{q}{4} dZ d\overline{Z} + \sum_{r.s=1}^{q-1} \tilde{g}_{rs}\, dw_r d\overline{w}_s \tag{6.35}$$

where the coefficient of $dZd\overline{Z}$ is constant. Invariance of the metric under a translation of all the vortices implies that

$$\sum_r b_r = 0 \tag{6.36}$$

and rotational invariance implies that

$$\sum_r \overline{Z}_r b_r \tag{6.37}$$

(details in [75]).

6.4 Asymptotic form of the metric

Following [76] we will compute the two vortex metric for the case of well separated vortices. To use the Samols formula, we need to compute the functions b_s. Let us denote the positions of the two vortices by Z_1 and Z_2. Working in the centre of mass frame

$$\begin{aligned} Z_1 &= re^{\imath\theta} && (6.38a) \\ Z_2 &= -re^{\imath\theta} && (6.38b) \end{aligned}$$

From the symmetry under rotations around the centre of mass, and the symmetry properties mentioned above, we have

$$b_1 = -b_2 = b(r)e^{\imath\theta} \tag{6.39}$$

The moduli space metric for this case, is reduced to

$$ds^2 = \frac{1}{2}\left[1 + \frac{1}{r}\frac{d}{dr}\left(rb(r)\right)\right](dr^2 + r^2\, d^2\theta) \tag{6.40}$$

Our task is reduced to obtain $b(r)$ for two asymptotically separated vortices. The procedure to compute $b(r)$ comes in two steps. First we solve the one vortex equation at large distances from the vortex origin

$$\frac{d^2 h_0}{dr^2} + \frac{1}{r}\frac{dh_0}{dr} - e^{h_0} + 1 = \tag{6.41}$$

this is the same computation done by Nielsen and Olesen [61] described in section 3.2. The solution is the modified Bessel function of zero order

$$h_0(r) \sim \frac{A}{\pi} K_0(r) \tag{6.42}$$

Where A is a constant, that recently has been determined by an argument involving dualities in string theory [73] to be $A = -2\pi\sqrt[4]{8}$.

Second we will study how h_0 is affected near its core by an asymptotically far away vortex. In this case the equations of motion can be linearised, and we have $h = h_0 + h_1$ where h_1 represents the effect of the far away vortex near the core of the first vortex. The linearization of the vortex equation yields

$$\left(\Delta - e^{h_0}\right) h_1 = 0 \tag{6.43}$$

The singularity of h at the core of the first vortex, is entirely carried by the function h_0, since the operator acting on h_1 is smooth at the core of the first vortex. Since h_0 is circularly symmetric, we can separate variables, and make a Fourier decomposition of $h_1(r,\theta)$

$$\begin{aligned} h(r,\theta) &= h_0(r) + h_1(r,\theta) \\ &= h_0(r) + \frac{1}{2} f_0(r) + \sum_{n=1}^{\infty} f_n(r)\cos n\theta + g_n(r)\sin n\theta \end{aligned} \tag{6.44}$$

where f_n obey the equation

$$\frac{d^2 f_n}{dr^2} + \frac{1}{r}\frac{df_n}{dr} - f_n\left(e^{h_0} + \frac{n^2}{r^2}\right) = 0 \tag{6.45}$$

and the same equation is obeyed by the functions g_n. Under a reflection along the X axis (i.e. the change $\theta \to -\theta$), h is invariant, meaning that the functions g_n are all zero. The functions f_n are all regular, and have a series expansion in powers of r

$$f_n = \alpha_n r^n + \dots \tag{6.46}$$

By identifying the terms linear in r, we can see that

$$b_1 = \alpha_1 \tag{6.47}$$

We will not tell here the details of the computation of α_1, the interested reader should consult [76], but simply say that the result that determines the asymptotic metric is

$$b(r) = \frac{A^2}{2\pi^2} K_1(2r) \tag{6.48}$$

The form of the asymptotic metric is given by

$$ds^2 = 2\pi\left[1 - \frac{A^2}{\pi^2} K_0(2r)\right](dr^2 + r^2\, d^2\theta) \tag{6.49}$$

The computation can be generalised to the case of q different, asymptotically separated vortices

$$ds^2 = \sum_{r\neq s} \pi\left[\frac{1}{2q} - \frac{A^2}{4\pi^2} K_0(|Z_{rs}|)\right] dZ_{rs} d\overline{Z}_{rs} \tag{6.50}$$

6.5 The metric for $N = 2$ semilocal strings

Here we will follow the steps of [77], where the metric of $N = 2$ semilocal strings is computed. In this work the two vortex metric is computed numerically. Some properties of the metric, like being kähler is proved, and the asymptotic form of the metric is computed. We will only sketch the main result of this work, and ask the interested reader to consult the original reference.

The kinetic energy for the $N = 2$ extended abelian Higgs model in the $A_0 = 0$ gauge is given by

$$T = \frac{1}{2}\int d^2x \left(\dot{\phi}^+\phi + \dot{A}_i^2\right) \tag{6.51}$$

as in the previous case, we only have to parametrise the gauge and Higgs fields in some coordinates of the moduli space, and the kinetic energy will automatically yield a quadratic expression in the time derivatives of these coordinates. As coordinates for the moduli, we can choose the coefficients of the polynomials $P(z)$ and $Q(z)$, that we named p_k and q_k, the more physical coordinates Z_r and λ_r that can be interpreted as the positions and sizes of the vortices when they are one far one from the other, or the "mixed" coordinates, where we label the position in the moduli space by the positions of the zeroes of ϕ_1, and the coefficients of the Polynomial $P(z)$. These mixed coordinates are the most convenient.

As usual we have to add the Gauss law

$$\partial_i \dot{A}_i = \imath \mathrm{Im}(\phi^+\phi) \tag{6.52}$$

For points different from the zeros of the Higgs field ϕ_1, we have

$$\dot{A} = \imath\partial\overline{\eta} \tag{6.53}$$

where η is given by

$$\eta = \frac{\dot{\phi}_1}{\phi_1} \tag{6.54}$$

from equation (3.100), we have

$$\dot{\phi}_2 = \phi_1 w + \phi_2 \eta \tag{6.55}$$

where $w = \phi_2/\phi_1$. The time derivative of the vortex equation for the extended abelian Higgs model, and the Gauss law can be seen as the real and imaginary part of

$$\Delta\eta - e^f(1 + |w|^2)\eta - \overline{w}\dot{w}e^f = 0 \tag{6.56}$$

where $f = \ln|\phi_1|^2$. The boundary conditions for η can be obtained by requiring the finiteness of the kinetic energy

$$\eta \overset{|z|\to\infty}{\longrightarrow} 0 \tag{6.57}$$

Equation (6.56) can be extended to all the complex plane by noting that near a zero of ϕ_1

$$\eta = \frac{-\dot{Z}_r}{z - Z_r} \tag{6.58}$$

so that

$$\Delta\eta - e^f(1 + |w|^2)\eta - \overline{w}\dot{w}e^f = -4\pi\sum_{r=1}^{q} \dot{Z}_r \partial\delta(z - Z_r) \tag{6.59}$$

Now we are in position of computing the metric. Substituting all the terms in the kinetic energy, we obtain

$$T = \frac{1}{2}\int d^2x \left\{\partial(\overline{\eta}\overline{\partial}\eta) + e^f\dot{\overline{w}}(\dot{w} + w\eta)\right\} \tag{6.60}$$

We will divide the integral in two parts. One what we will call D is a union of discs D_r, of radius ε centred at the zeros of the Higgs field Z_r. It is clear that choosing ε small enough we can always make all the discs D_r non overlapping. The second part is the contribution for the rest of the space $\mathbb{R}^2 - D$. The integral over D vanish in the limit $\varepsilon \to 0$, whereas in $\mathbb{R}^2 - D$, η is a smooth function. Using equation (6.58) and

$$\eta = \frac{\partial f}{\partial Z_r}\dot{Z}_r + \frac{\partial f}{\partial q_r}\dot{q}_r \tag{6.61}$$

the first term in the integral gives

$$\int_0^{2\pi} d\theta_s \overline{\partial}\left(\frac{\partial f}{\partial Z_r}\dot{Z}_r + \frac{\partial f}{\partial q_r}\dot{q}_r\right)\dot{\overline{Z}}_s \tag{6.62}$$

where for each s the integration is performed in a circle of radious ε, and centered at Z_r. If we make a Taylor expansion of the smooth part of f near Z_s, we obtain

$$\begin{aligned} f &= \ln|z - Z_s|^2 + a_s + \frac{1}{2}\left[b_s(z - Z_s) + \overline{b}_s(\overline{z} - \overline{Z}_s)\right] \\ &+ c_s(z - Z_s)^2 + \overline{c}_s(\overline{z} - \overline{Z}_s)^2 + d_s(z - Z_s)(\overline{z} - \overline{Z}_s) \\ &+ \mathcal{O}(\varepsilon^3) \end{aligned} \tag{6.63}$$

The vortex equation of the extended abelian Higgs model (equation (3.109)) requires

$$d_s = -\frac{1}{4}\left(1 - |\lambda_s|^2 e^{a_s}\right) \tag{6.64}$$

where λ_s are defined by equation (3.125). This means that for z close to Z_s

$$\overline{\partial}\left(\frac{\partial f}{\partial Z_r}\right) = \frac{1}{2}\frac{\partial \overline{b}_s}{\partial Z_r} - d_r\delta_{rs} + \mathcal{O}(\varepsilon) \tag{6.65a}$$

$$\overline{\partial}\left(\frac{\partial f}{\partial q_k}\right) = \frac{1}{2}\frac{\partial \overline{b}_s}{\partial q_k} + \mathcal{O}(\varepsilon) \tag{6.65b}$$

Taking the limit $\varepsilon \to 0$, the expression for the kinetic energy is

$$\begin{aligned} T &= \frac{\pi}{2}\left\{\frac{1}{4}\left(1 - |\lambda_s|^2 e^{a_s}\right)\delta_{rs} + 2\frac{\partial\overline{b}_s}{\partial Z_r} + \frac{1}{\pi}\int d^2x\,\frac{\partial}{\partial Z_r}(we^f)\frac{\partial\overline{w}}{\partial\overline{Z}_s}\right\}\dot{Z}_r\dot{\overline{Z}}_s \\ &+ \frac{\pi}{2}\left\{2\frac{\partial\overline{b}_s}{\partial q_k} + \frac{1}{\pi}\int d^2x\,\frac{\partial}{\partial q_k}(we^f)\frac{\partial\overline{w}}{\partial\overline{Z}_s}\right\}\dot{q}_k\dot{\overline{Z}}_s \\ &+ \frac{1}{2}\left\{\int d^2x\,\frac{\partial}{\partial Z_r}(we^f)\frac{\partial\overline{w}}{\partial\overline{q}_k}\right\}\dot{Z}_r\dot{\overline{q}}_k \\ &+ \frac{1}{2}\left\{\int d^2x\,\frac{\partial}{\partial q_k}(we^f)\frac{\partial\overline{w}}{\partial\overline{q}_s}\right\}\dot{q}_k\dot{\overline{q}}_s \end{aligned} \tag{6.66}$$

$$\tag{6.67}$$

Because of the slow fall-off of w at infinity, the coefficient of the $\dot{q}_{q-1}\dot{\overline{q}}_{q-1}$ diverges at infinity. This means that finitness of the kinetic energy requires

$$q_{q-1} = \sum_r \lambda_r = 0 \tag{6.68}$$

this makes that the terms proportional to $\dot{q}_{q-1}$ may be excluded form the expression of the energy. The function f has singularities at Z_r. It is convenient to introduce a quantity

$$\Phi = f - \ln |P(z)|^2 \tag{6.69}$$

that is smooth everywhere. We can now define

$$\tilde{b}_s = b_s - \sum_{r \neq s} \frac{1}{z - Z_r} \tag{6.70}$$

where

$$\tilde{b}_s = \partial \Phi(Z_s) \tag{6.71}$$

is smooth. The metric can be rewritten in a more appropiate way with the help of the definitions

$$B_r = \partial \Phi(Z_r) + \frac{1}{2\pi} \int d^2x \, |P|^2 e^{\Phi} \frac{\partial |w|^2}{\partial Z_r} \tag{6.72a}$$

$$C_k = \frac{1}{2\pi} \int d^2x \, |P|^2 e^{\Phi} \frac{\partial |w|^2}{\partial q_k} \tag{6.72b}$$

the metric can be written

$$ds^2 = \frac{1}{2} \left(\delta_{rs} + 2 \frac{\partial \overline{B}_s}{\partial Z_r} \right) dZ_r d\overline{Z_s} + \frac{\partial \overline{B}_s}{\partial q_k} dq_k d\overline{Z}_s + \frac{\partial \overline{C}_s}{\partial Z_r} dZ_r d\overline{q}_s + \frac{\partial \overline{C}_s}{\partial q_k} dq_k d\overline{q}_s \tag{6.73}$$

This is the final expression that we were looking for. Since the energy is real, we have the following relations between the coefficients

$$\frac{\partial \overline{B}_s}{\partial Z_r} = \frac{\partial B_r}{\partial \overline{Z}_s} \tag{6.74a}$$

$$\frac{\partial \overline{B}_s}{\partial q_k} = \frac{\partial C_k}{\partial \overline{Z}_s} \tag{6.74b}$$

$$\frac{\partial \overline{C}_l}{\partial q_k} = \frac{\partial C_k}{\partial \overline{q}_l} \tag{6.74c}$$

As for standard vortices the metric is Kähler (for a proof see [77]). In general it is difficult to obtain the coefficients B_r and C_k that are needed to compute the metric, and numerical tools should be used, but some general properties of the metric, like its invariance under translations and rotations can be deduced without an explicit computation of these coefficients. The interested reader should consult [77] for details.

Seven

The metric for vortices on the Torus

7.1 Derivation of the metric in the Torus

In the previous chapter we have shown the Samols method to compute the metric of the vortices. Here we will show another way to compute the metric of the moduli space of vortices living in the Torus.

To obtain the metric we have to compute the kinetic energy, which in the $A_0 = 0$ gauge, takes the form:

$$T = \frac{1}{2}\int_{\mathbb{T}^2} d^2x\,(\dot{A}_i^2 + |\dot{\phi}|^2) \tag{7.1}$$

In principle we only have to parametrise A_i and ϕ and their time derivatives in terms of whatever coordinates of the moduli space we choose, and plug them into the expression for T. To do that, first of all, we will use our usual parametrisation for the fields with our complex notation (see section (A.4))

$$\begin{aligned} A &= -\imath\partial\overline{H} + v - \imath f\overline{z} && (7.2a)\\ \phi &= \sqrt{\epsilon}e^{-H}\chi^{(v)} && (7.2b) \end{aligned}$$

taking the time derivative, we have

$$\begin{aligned} \dot{A} &= -\imath\partial\dot{\overline{H}} && (7.3a)\\ \dot{\phi} &= \sqrt{\epsilon}e^{-H}\left(\dot{\chi}^{(v)} - \dot{H}\chi^{(v)}\right) && (7.3b) \end{aligned}$$

The equations that we have to solve to obtain the functions $\chi^{(v)}$ and H are the Bogomolny equations (3.54a), and to determine the time dependence of these functions we need the time derivative of the Bogomolny equations. In total these four equations are

$$\begin{aligned} (\overline{\partial} + fz)\chi^{(v)} &= \imath\overline{v}\chi^{(v)} && (7.4a)\\ \partial\overline{\partial}h &= \frac{\epsilon}{2}\left(1 - e^{-2h}|\chi^{(v)}|^2\right) && (7.4b)\\ (\overline{\partial} + fz)\dot{\chi}^{(v)} &= \imath(\bar{v}\dot{\chi}^{(v)} + \dot{\bar{v}}\chi^{(v)}) && (7.4c)\\ \partial\overline{\partial}\dot{h} &= \frac{\epsilon}{2}(2\dot{h}|\phi|^2 - e^{-2h}\frac{\partial}{\partial t}|\chi^{(v)}|^2) && (7.4d) \end{aligned}$$

One has to add the Gauss constraint (one of Maxwell equations for the system)

$$\partial_i \dot{A}_i = \mathrm{Im}(\dot{\phi}\overline{\phi}) \tag{7.4e}$$

This can be combined with the time derivative of the second Bogomolny equation to give a complex equation:

$$\partial \dot{\overline{A}} = -\imath \overline{\phi}\dot{\phi} \tag{7.4f}$$

that can be written

$$\hat{O}\dot{H} = \epsilon e^{-2h} \overline{\chi}^{(v)} \dot{\chi}^{(v)} \tag{7.4g}$$

Where $\hat{O} \equiv -\partial\overline{\partial} + |\phi|^2$. We know that the field $\chi^{(v)}$ contains all the information about the moduli space, and can be explicitly written as

$$\chi^{(v)} = \sum_s c_s \left(\mathcal{T}_v\right)^{-1} [\chi_s] \tag{7.5}$$

where c_s are q arbitrary complex constants and the operator $\left(\mathcal{T}_v\right)^{-1}$ can be computed from equation (4.26), and acts in the following way

$$\left(\mathcal{T}_v\right)^{-1} [\chi_s] = e^{\frac{|v|^2}{4f}} e^{\frac{\imath}{2}(vz + \overline{v}\overline{z})} \chi_s \left(z - \frac{\imath \overline{v}}{2f}, \overline{z} + \frac{\imath v}{2f}\right) \tag{7.6}$$

the functions χ_s are known functions (the Theta functions with characteristics (see appendix B)), and are given by the "ground state" functions of the appendix C (look at the definition C.1.1)

$$\chi_s = | \, 0, s\rangle \tag{7.7}$$

Our parametrisation of the fields automatically solves equations (7.4a,7.4c), so in our parametrisation, the only equations that remains unsolved are:

$$\partial\overline{\partial}h = \frac{\epsilon}{2}\left(1 - e^{-2h}\overline{\chi}^{(v)}\chi^{(v)}\right) \tag{7.8a}$$

$$\hat{O}\dot{H} = \epsilon e^{-2h}\overline{\chi}^{(v)}\dot{\chi}^{(v)} \tag{7.8b}$$

Integrating both sides of Eqs. 7.8, we obtain

$$\frac{1}{\mathcal{A}}\int d^2x \, e^{-2h}\overline{\chi}^{(v)}\chi^{(v)} = 1 \tag{7.9a}$$

$$\int d^2x \, e^{-2h}\overline{\chi}^{(v)}\dot{\chi}^{(v)} = \int d^2x \, e^{-2h}\overline{\chi}^{(v)}\dot{H}\chi^{(v)} \tag{7.9b}$$

that are basic equations for the constant pieces of h and $\dot{H}$. Equations (7.8,7.9) determine uniquely h and $\dot{H}$. Substituting these expressions in the equation for the metric we arrive at

$$\begin{aligned} T &= \frac{1}{2}\int d^2x \left\{|\dot{v}|^2 + \epsilon e^{-2h}\left|\dot{\chi}^{(v)}\right|^2 - \dot{\overline{H}}\hat{O}\dot{H}\right\} \\ &= \frac{1}{2}\int d^2x \left\{|\dot{v}|^2 + \epsilon e^{-2h}\dot{\overline{\chi}}^{(v)}\left(\dot{\chi}^{(v)} - \dot{H}\chi^{(v)}\right)\right\} \end{aligned} \tag{7.10}$$

This is the crucial expression from which the metric and its perturbative expansion can be obtained. It is easy to understand that if we obtain the functions h and $\dot{H}$ as a power series in our Bradlow parameter, the previous formula will induce also a power series for the metric in the Bradlow parameter, and this is the central idea. The function h can be obtained as we have done to obtaining the solutions of the vortex equation, and the equation for $\dot{H}$ can be solved in a similar way, as we will see.

7.1.1 Factorisation of the metric

In Eq. 7.10 the time derivatives of the coordinates appear in $\dot{v}$ and $\dot{\chi}^{(v)}$. These two terms are not independent variations within our fibre bundle choice of coordinates. One should rather use $\dot{v}$ and $\dot{\chi}$. For that purpose we need the formula

$$\dot{\chi}^{(v)} = \frac{d}{dt}\left((\mathcal{T}_v)^{-1}[\chi]\right) = (\mathcal{T}_v)^{-1}[\dot{\chi} - \frac{i\dot{\overline{v}}}{2f}(\partial - f\overline{z} + iv)\chi] \tag{7.11}$$

Using this expression we might also relate the solution of Eq. 7.8b for $v \neq 0$, which we will note $\dot{H}_v$, with the corresponding one for $v = 0$, that we will call simply $\dot{H}$:

$$\dot{H}_v\left(z + \frac{i\overline{v}}{2f}, \overline{z} - \frac{iv}{2f}\right) = \dot{H}(z, \overline{z}) - \frac{i\dot{\overline{v}}}{2f}(\delta(z, \overline{z}) + iv) \tag{7.12}$$

where δ is the solution of the equation

$$\hat{O}\delta = \epsilon e^{-2h}\overline{\chi}(\partial - f\overline{z})\chi \tag{7.13}$$

Now we should plug these relations into the formula for the kinetic term T, and after changing variables in the integrals and decomposing χ in the aforementioned basis, we arrive at an expression of the form

$$T = \frac{\pi}{2}\left(g_{vv}(c_i)\dot{v}\dot{\overline{v}} + g_{iv}(c_i)\dot{\overline{c}}_i\dot{\overline{v}} + g_{vi}(c_i)\dot{v}\dot{c}_i + g_{ij}(c_i)\dot{c}_i\dot{\overline{c}}_j\right) \tag{7.14}$$

Where $g_{\alpha\beta}$ is the induced metric in the moduli space (we follow the normalisation of Ref. [78]). We can show that g_{iv} and g_{vi} vanish. To show this we focus on the term proportional to $\dot{v}$ in the expression of T. It has the form

$$\frac{i\dot{v}}{4f}\int d^2x\, I \tag{7.15}$$

where

$$I = (\overline{\partial} - fz)\overline{\chi}\left[\dot{\chi} - \chi\dot{H}\right]\epsilon e^{-2h} = -\overline{\partial}\partial\overline{\partial}\dot{H} + \epsilon\overline{\partial}\dot{H} - 2\partial(\overline{\partial}\dot{H}\,\overline{\partial}h) \tag{7.16}$$

This, being the derivative of an analytic periodic function, has a vanishing integral. By hermiticity the same holds for the term proportional to $\dot{\overline{v}}$. In a similar fashion (by partial integration), one can show that the term proportional to $|\dot{v}|^2$ is a constant. Finally, we arrive to the following expression for the metric

$$ds^2 = \frac{\mathcal{A}^2}{4\pi^2 q}\,dv d\overline{v} + \frac{\epsilon}{\pi}\int d^2x\left\{e^{-2h}\dot{\overline{\chi}}\left(\dot{\chi} - \dot{H}\chi\right)\right\} \tag{7.17}$$

The first term is the metric associated to the dual torus, while the second is the metric of the fibre $\mathcal{F}$. This factorised structure was previously found in Ref. [72].

From now on we might focus upon the metric of the fibre $\mathcal{F}$, which can be written as

$$ds^2 = \frac{\epsilon}{\pi}\int d^2x\, e^{-2h}\dot{\overline{\chi}}\left(\dot{\chi} - \dot{H}\chi\right) = \frac{-1}{\pi}\int d^2x\,\partial\left(\frac{\dot{\overline{\chi}}}{\overline{\chi}}\overline{\partial}\dot{H}\right) \tag{7.18}$$

Notice that, this time, the integral of a partial derivative does not vanish, because the quantity inside parenthesis has singularities (poles) at the zeroes of $\overline{\chi}$. Hence, the metric is expressed by the value of this function in the vicinity of these zeroes. This property is similar to the Samols expression for computing the metric, that we have explained in the section 6.2.

7.2 Symmetries and general properties of the metric

If we compute the metric in homogeneous coordinates, we have

$$T = g_{ij}(c)\dot{\bar{c}}_i c_j \tag{7.19}$$

since multiplying all the c_i by a time independent factor λ cannot change the value of the energy, because it represent the same point in the moduli space, we obtain

$$g_{ij}(\lambda c) = \frac{1}{\lambda^2} g_{ij}(c) \tag{7.20}$$

but we can also make this factor depend on time, and for the same reason, this cannot change the value of the energy. This means that

$$g_{ij} c_j = \bar{c}_j g_{ji} = 0 \tag{7.21}$$

A symmetry of our system is the rotation of 180 degrees. This rotation can change the position of the zeros, and will represent in general other point in the moduli space, but the value of the energy cannot change. Under such a rotation, the homogeneous coordinates change as follows

$$c_s \longrightarrow R_{ss'} c_{s'} \tag{7.22}$$

where the $q \times q$ matrix $R_{ss'}$ is given by

$$R_{ss'} = \begin{cases} 1 & s + s' = 0 \pmod q \\ 0 & \text{any other case} \end{cases} \tag{7.23}$$

7.3 Different coordinates in the moduli

Here we recover the expression for the metric of the fibre $\mathcal{F}$:

$$ds^2 = \frac{\epsilon}{\pi} \int d^2x\, e^{-2h} \dot{\bar{\chi}} \left(\dot{\chi} - \dot{H}\chi \right) \tag{7.24}$$

To obtain the metric in whatever coordinates of the fibre that we want, we only have to parametrise χ in terms of such coordinates. The most natural option are the homogeneous coordinates c_i. In this coordinates

$$\chi = \sum c_s \chi_s \tag{7.25}$$

where the functions $\chi_s \equiv \mid 0, s\rangle$ are given in the appendix C. The expression for the metric in these coordinates is

$$ds^2 = g_{ij}\, d\bar{c}_i dc_j = \frac{\epsilon}{\pi} \int d^2x\, e^{-2h} \overline{\chi_i} \left(\chi_j - \mathcal{H}_j \chi \right) d\bar{c}_i dc_j \tag{7.26}$$

where $\mathcal{H}_j \dot{c}_j = \dot{H}$.

Once we have the metric in the homogeneous basis, we can compute the form of the metric in any other basis by a change of coordinates. First we can use as inhomogeneous basis the coordinates $Z_i = c_{i+1}/c_1$ with $(i = 1, \ldots, q-1)$. Writing

$$\begin{pmatrix} dc_1 \\ dc_2 \\ \vdots \\ dc_q \end{pmatrix} = dc_1 \begin{pmatrix} 1 \\ Z_1 \\ \vdots \\ Z_{q-1} \end{pmatrix} + c_1 \begin{pmatrix} 0 \\ dZ_1 \\ \vdots \\ dZ_{q-1} \end{pmatrix} \tag{7.27}$$

and using the symmetry property (7.21) it is easy to prove that

$$g_{i1} + g_{i,j+1} Z_j = 0 \tag{7.28}$$

and then, the metric cancels the first term of equation (7.27). These means that we can write the metric as

$$ds^2 = |c_1|^2 g_{i+1,j+1} d\overline{Z}_i dZ_j \tag{7.29}$$

and now using that

$$|c_1|^2 = \frac{\|c\|^2}{1 + \sum_i |Z_i|^2} \tag{7.30}$$

where $\|c\|^2 = \sum_i \|c_i\|^2$, we can write the metric as

$$ds^2 = |c|^2 \frac{g_{i+1,j+1}}{1 + \sum_i |Z_i|^2} d\overline{Z}_i dZ_j \tag{7.31}$$

A very natural basis is the one in which the coordinates are the position of the zeros of the Higgs field. The change of basis function to obtain the metric in this basis is given by

$$Z_s = \frac{\varepsilon_{s+1,i_1 \ldots i_{q-1}} \prod_l \chi_{i_l}(\omega_l)}{\varepsilon_{1,i_1 \ldots i_{q-1}} \prod_l \chi_{i_l}(\omega_l)} \tag{7.32}$$

where $\{\omega_i\}$ are the position of the $|q|$ zeros of the Higgs field. We can write

$$dZ_s = A_{sj} d\omega_j \tag{7.33}$$

where

$$A_{sj} = Z_s \left\{ \frac{\varepsilon_{s+1,i_1 \ldots i_{q-1}}}{\varepsilon_{s+1,i_1 \ldots i_{q-1}} \prod_l \chi_{i_l}(\omega_l)} - \frac{\varepsilon_{1,i_1 \ldots i_{q-1}}}{\varepsilon_{1,i_1 \ldots i_{q-1}} \prod_l \chi_{i_l}(\omega_l)} \right\} \sum_k \left(\prod_{l \neq k} \chi_{i_l}(\omega_l) \right) \chi'_{i_k}(\omega_k) \tag{7.34}$$

and the final formula for the metric is given by

$$ds^2 = \|c\|^2 \frac{g_{i+1,j+1}}{1 + \sum_i |Z_i|^2} \overline{A}_{il} A_{jk} \, d\overline{\omega}_l d\omega_k \tag{7.35}$$

As we will see, these formulas are very simple for the special case of the two vortex dynamics.

7.3.1 Two vortex dynamics

Since the vortex are localised objects, the most important case is the two vortex dynamics, and it is this case the one that we are going to study in more detail. We will start writing the metric in the homogeneous basis

$$ds^2 = g_{ij} d\overline{c}_i dc_j \tag{7.36}$$

Now we will change to the non-homogeneous basis, using the same procedure as below

$$Z = \frac{c_2}{c_1}; \quad \overline{Z} = \frac{\overline{c}_2}{\overline{c}_1} \tag{7.37}$$

In this basis the metric takes the form

$$ds^2 = \frac{f(Z, \overline{Z})}{(1 + |Z|^2)^2} d\overline{Z} dZ \tag{7.38}$$

where the function $f(Z,\overline{Z}) = \|c\|^2\mathrm{Tr}(g) = \|c\|^2(g_{11}+g_{22})$. If we split Z in its real and imaginary parts

$$Z = a + \imath b \tag{7.39}$$

we can write the metric in term of the new coordinates (θ,ψ) defined by[1]

$$\frac{2a}{1+|Z|^2} = \sin\theta\cos\psi \tag{7.40a}$$

$$\frac{1-|Z|^2}{1+|Z|^2} = \sin\theta\sin\psi \tag{7.40b}$$

$$\frac{-2b}{1+|Z|^2} = \cos\theta \tag{7.40c}$$

and the metric takes the form

$$ds^2 = \frac{f(\theta,\psi)}{4}\left(d\theta^2 + \sin^2\theta\, d\psi^2\right) \tag{7.41}$$

In the basis of the zeros of the Higgs field, the general formula for the change of basis function is simplified. If one of the zeros of the Higgs field is located in the point u (the position of the other zero is given by the centre of mass condition), defined by

$$u = \frac{x_1 + \imath x_2}{2} \tag{7.42}$$

the *relative* coordinate is given by

$$w = \frac{2\pi q}{l_1}u \tag{7.43}$$

using the change of basis function given in Ref. [71] it is easy to write the coordinate Z as a function of w

$$Z(w) = -\frac{\vartheta_3(w|\imath q\tau)}{\vartheta_2(w|\imath q\tau)} \tag{7.44}$$

where $\vartheta_i(z|t)$ are the i^{th} classical Jacobi theta functions (For an explicit expression of these functions, and an explanation of the most important properties of these functions, see the appendix B) and $\tau = l_2/l_1$ (when we do not write it explicitly, we will assume that the period of all the theta functions is $\imath q\tau$). Finally the metric in the relative coordinates is given by

$$ds^2 = f(w,\overline{w})|\vartheta_4(0)|^4\frac{|\vartheta_1(w)|^2|\vartheta_4(w)|^2}{\left(|\vartheta_2(w)|^2 + |\vartheta_3(w)|^2\right)^2}\,d\overline{w}dw \tag{7.45}$$

in the coordinates $(u,\overline{u})$ the metric is given by

$$ds^2 = f(u,\overline{u})\left(\frac{2\pi q}{l_1}\right)^2|\vartheta_4(0)|^4\frac{\left|\vartheta_1\left(\frac{2\pi q u}{l_1}\right)\right|^2\left|\vartheta_4\left(\frac{2\pi q u}{l_1}\right)\right|^2}{\left(\left|\vartheta_2\left(\frac{2\pi q u}{l_1}\right)\right|^2 + \left|\vartheta_3\left(\frac{2\pi q u}{l_1}\right)\right|^2\right)^2}\,d\overline{u}du \tag{7.46}$$

[1]These is very special of the $q=2$ case, since here the moduli space $\mathbb{CP}^1 \approx S_2$, and then we can use any coordinates of the two sphere.

7.4 The metric of the moduli for semi local strings

In this section we will obtain the metric in the moduli space for the case of semilocal strings. In this case, the kinetic energy in the $A_0 = 0$ gauge is given by

$$T = \int_{\mathbb{T}^2} d^2x \left(\dot{A}_i + |\dot{\phi}^{(a)}|^2 \right) \tag{7.47}$$

as we have seen the moduli space of solutions of the Bogomolny equations of the extended abelian Higgs model has a fibre bundle structure, with base space parametrised by the complex constant v, that lives in the dual torus, and a fibre that is the complex projective space $\mathbb{CP}^{Nq-1}$. The $N \times q$ complex homogeneous coordinates $c_i^{(a)}$ $(i = 1, \ldots, q; a = 1, \ldots, N)$ of the fibre provides a way of studying the form of the solutions, as we have done in the section 5.3. Just as in the case of the non extended abelian Higgs model, the kinetic energy induces a metric in the moduli space whose geodesics determine the low energy scattering of these semilocal strings, and we will be interested in the part of the metric that corresponds to the fibre, since the dependence on the coordinate v can be eliminated working in the centre of mass frame.

$$T = \frac{\pi}{2} g_{ij}^{(a,b)} \dot{c}_i^{(a)} \dot{\bar{c}}_j^{(b)} \tag{7.48}$$

again we will use complex notation A.4, and the parametrisation of the fields

$$\begin{aligned} A &= -\imath \partial H + v - \imath f \bar{z} && (7.49a) \\ \phi_a &= \sqrt{\epsilon} e^{-H} \chi_a^{(v)} && (7.49b) \end{aligned}$$

The equations that we have to solve to obtain the functions $\chi_a^{(v)}$ and H and its time dependence are the Bogomolny equations, its time derivative and the Gauss law

$$\begin{aligned} (\overline{\partial} + fz)\chi_a^{(v)} &= \imath \bar{v} \chi_a^{(v)} && (7.50a) \\ \partial\overline{\partial} h &= \frac{\epsilon}{2}\left(1 - e^{-2h}|\chi_a^{(v)}|^2\right) && (7.50b) \\ (\overline{\partial} + fz)\dot{\chi}_a^{(v)} &= \imath(\bar{v}\dot{\chi}_a^{(v)} + \dot{\bar{v}}\chi_a^{(v)}) && (7.50c) \\ \partial\overline{\partial}\dot{h} &= \frac{\epsilon}{2}(2\dot{h}|\phi_a|^2 - e^{-2h}\frac{\partial}{\partial t}|\chi_a^{(v)}|^2) && (7.50d) \\ \partial_i \dot{A}_i &= \mathrm{Im}(\phi^+ \dot{\phi}) && (7.50e) \end{aligned}$$

The Higgs field can be written in terms of the coordinates of the fibre $c_i^{(a)}$

$$\chi_a^{(v)} = \sum_s c_s^{(a)} (\mathcal{T}_v)^{-1}[\chi_s] \tag{7.51}$$

where the operator $\mathcal{T}_v$ is defined in equation (4.26) (see also equation (7.6)), and the functions χ_s are defined in the appendix C in terms of the Theta functions with characteristics. This parametrisation automatically solves equations (7.50a) and (7.50c). Equation (7.50b), is the vortex equation, and equations (7.50d) and (7.50e) can be combined together in the equation

$$\hat{O}\dot{H} = \epsilon e^{-2h} \overline{\chi}_a^{(v)} \dot{\chi}_a^{(v)} \tag{7.52}$$

where the operator $\hat{O}$ is given by

$$\hat{O} = -\partial\overline{\partial} + |\phi_a|^2 \tag{7.53}$$

In what follows we will work in the centre of mass frame. We will define $\mathcal{H}_{i(a)}$ with the formula

$$\dot{H} = \mathcal{H}_{i(a)}\dot{c}_i^{(a)} \tag{7.54}$$

it is easy to see that the equation that obey $\mathcal{H}_{i(a)}$ is

$$\hat{O}\mathcal{H}_{s'(a)} = \epsilon e^{-2h}\overline{c}_s^{(a)}\overline{\chi}_s\chi_{s'} \tag{7.55}$$

Putting these pieces together we arrive to our final expression for the metric in the moduli space

$$g_{ij}^{(a,b)} = \frac{\epsilon}{\pi}\int d^2x\, e^{-2h}\left(\chi_i\delta_{ab} - c_k^{(a)}\chi_k\mathcal{H}_{i(b)}\right)\overline{\chi}_j \tag{7.56}$$

To compute the metric, we need both the function h and the functions $\mathcal{H}_{i(a)}$. The first of these equations can be obtained by solving the vortex equation, and the rest of the functions $\mathcal{H}_{i(a)}$ should be obtained by solving the equations (7.55). This is the fundamental expression that we will use to compute the metric.

It is important to note here that the relation between the homogeneous coordinates of the moduli and the physical parameters that characterise a configuration is not clear. For the case of the non extended abelian Higgs model there is a one to one correspondence between the homogeneous coordinates of the moduli and the *position* of the vortex strings, but for the extended abelian Higgs model, this relation is no longer valid. We have seen $q = 2$ configurations with four structures, and configurations with different widths.

There is a convenient way of writing the metric. Under a $SU(N)$ rotation, the metric transforms as follow

$$g_{ij}^{(a,b)} \longrightarrow \Omega_{aa'}g_{ij}^{(a',b')}\overline{\Omega}_{bb'} \tag{7.57}$$

this means that we can write the metric in the following way

$$g_{ij}^{(a,b)} = c_{i'}^{(a)}\tilde{g}_{i'ijj'}\overline{c}_{j'}^{(b)} \tag{7.58}$$

where the function $\tilde{g}_{i'ijj'}$ is now invariant under a $SU(N)$ transformation. Being an $SU(N)$ invariant, means that $\tilde{g}_{i'ijj'}$ only depends of the invariant matrix $M = c^+c$, so we can write

$$g_{ij}^{(a,b)} = c_{i'}^{(a)}\tilde{g}_{i'ijj'}(M)\overline{c}_{j'}^{(b)} \tag{7.59}$$

Again, the metric only depends on all the coordinates of the moduli in a rather trivial way.The difficult dependence comes from the $SU(N)$ invariant quantity.

Eight

The Bradlow expansion for the metric

8.1 An expansion for the metric

We have an expression for the metric in the homogeneous coordinates in terms of $q+1$ unknown functions h and $\mathcal{H}_i$

$$g_{ij}(c_k) = \frac{\epsilon}{\pi}\int d^2x\, e^{-2h}\overline{\chi}_i\left(\chi_j - \mathcal{H}_j\chi\right) \tag{8.1}$$

The idea of the Bradlow parameter expansion is very easy. If we have the functions h and $\mathcal{H}_i$ as a power expansion in the Bradlow parameter ϵ, the previous expansion will induce a similar expansion for the metric

$$g_{ij} = \sum_n g_{ij}^{(n)}\epsilon^n \tag{8.2}$$

To obtain the functions h and $\mathcal{H}_i$ we have to solve the following equations

$$\partial\overline{\partial}h = \frac{\epsilon}{2}\left(1 - e^{-2h}\overline{\chi}\chi\right) \tag{8.3a}$$

$$\hat{O}\mathcal{H}_i = \epsilon e^{-2h}\overline{\chi}\chi_i \tag{8.3b}$$

the constant parts of these functions are determined by integrating both equations

$$\frac{1}{\mathcal{A}}\int d^2x\, e^{-2h}\overline{\chi}\chi = 1 \tag{8.3c}$$

$$\int d^2x\, e^{-2h}\overline{\chi}\chi_i = \int d^2x\, e^{-2h}\overline{\chi}\mathcal{H}_i\chi \tag{8.3d}$$

In the chapter 5 we show a method to solve the vortex equation order by order in the quantity ϵ, and the same method can be applied here to solve the equation for $\mathcal{H}_i$. Since $\mathcal{H}_i$ is a periodic function, we can make a Fourier expansion

$$\mathcal{H}_i = \sum_{\vec{n}}\mathcal{H}_i(\vec{n})e^{2\pi\imath\left(n_1\frac{x_1}{l_1}+n_2\frac{x_2}{l_2}\right)} \tag{8.4}$$

and we will expand each of the Fourier modes as a power series in the Bradlow parameter ϵ

$$\mathcal{H}_i(\vec{n}) = \sum_k \mathcal{H}_i^{(k)}(\vec{n})\epsilon^k \tag{8.5}$$

Equation (8.3b) can be written as

$$-\partial\bar{\partial}\mathcal{H}_i = \epsilon e^{-2h}\overline{\chi}\left(\chi_i - \mathcal{H}_i\chi\right) \tag{8.6}$$

The l.h.s of equation (8.6) can be written (in the Fourier basis) as[1]

$$(1-\epsilon)\xi(\vec{n})\mathcal{H}_i^{(k)}(\vec{n})\epsilon^k \tag{8.7}$$

and, the r.h.s of equation (8.6) is given by

$$\sum_m \left\{ \bar{c}_j \sum_{\vec{k}} P^{(m)}(\vec{k})L^{ij}(\vec{n}-\vec{k}) - \sum_{l=0}^{m}\sum_{\vec{k}_1,\vec{k}_2} \mathcal{H}_i^{(m-l)}(\vec{k}_1)L(\vec{k}_2-\vec{k}_1)P^{(l)}(\vec{n}-\vec{k}_2) \right\}\epsilon^m \tag{8.8}$$

where $P^{(k)}(\vec{n})$ are the Fourier modes of the $k^{\underline{\text{th}}}$ order expansion of ϵe^{-2h}. That is to say

$$\epsilon e^{-2h} = \sum_k \sum_{\vec{n}} P^{(k)}(\vec{n}) e^{2\pi\imath\left(n_1\frac{x_1}{l_1}+n_2\frac{x_2}{l_2}\right)}\epsilon^k \tag{8.9}$$

and can be obtained by making some convolutions from the Fourier coefficients of $h^{(k)}(\vec{n})$ (see equation(5.11)). The zero mode of equation (8.8) must be zero, and this can be seen as an equation for the constant part of $\mathcal{H}_i^{(k)}$. Equating equal ϵ-orders of the equation allows the determination of $\mathcal{H}_i^{(k)}(\vec{n})$ as a function of $P^{(l)}(\vec{n})$ and $\mathcal{H}_i^{(l)}(\vec{n})$ for $l = 0,\ldots,k-1$, so these gives us again a recursive process to obtain the functions $\mathcal{H}_i^{(k)}(\vec{n})$ order by order. Once this is achieved, we can introduce these quantities and obtain a ϵ-expansion for the metric

$$g_{ij}(c_k) = \sum_n g_{ij}^{(n)}(c_k)\epsilon^n \tag{8.10}$$

where each order of the metric is explicitly given by

$$g_{ij}^{(n)}(c_k) = \frac{\mathcal{A}}{\pi}\left\{ \sum_{\vec{k}} P^{(n)}(\vec{k})L^{ij}(-\vec{k}) - c_l \sum_{k=0}^{n}\sum_{\vec{k}_1,\vec{k}_2} P^{(k)}(\vec{k}_1)\mathcal{H}_j^{(n-k)}(\vec{k}_2-\vec{k}_1)L^{il}(-\vec{k}_2) \right\} \tag{8.11}$$

8.1.1 Zero order computation

To zero order in epsilon, the metric is zero, cause $P^{(0)}(\vec{n}) = 0$.

$$g_{ij}^{(0)}(c_k) = 0 \tag{8.12}$$

8.1.2 First order computation

To compute the metric to this order is enough to know $\mathcal{H}_i^{(0)}(\vec{n})$ and $h^{(0)}(\vec{n})$, and only the constant terms of it are different from zero. So we obtain

$$\mathcal{H}_i^{(0)}(0) = \frac{\bar{c}_i}{|c|^2} \tag{8.13a}$$

$$h^{(0)}(0) = \ln|c| \tag{8.13b}$$

which gives us the first order metric

$$g_{ij}^{(1)} = \frac{\mathcal{A}}{\pi|c|^2}\left(\delta_{ij} - \frac{c_i\bar{c}_j}{|c|^2}\right) \tag{8.14}$$

[1] We use the same definitions as in the first section of the chapter 5

8.1.3 Second order computation

To obtain the metric to this order, we need the functions $\dot{\mathcal{H}}_i^{(k)}(\vec{n})$ and $h^{(k)}(\vec{n})$, for $k = 0, 1$. The zero order of these functions was computed before, and the first orders are given by

$$h^{(1)}(\vec{n}) = \begin{cases} \frac{L(\vec{n})}{2|c|^2\xi(\vec{n})} & \vec{n} \neq 0 \\ -\frac{S}{2|c|^4} S & \vec{n} = 0 \end{cases} \tag{8.15a}$$

$$\mathcal{H}_i^{(1)}(\vec{n}) = \frac{\overline{c}_j}{|c|^2} \left(\delta_{il} - \frac{\overline{c}_i c_l}{|c|^2} \right) \frac{L^{jl}(\vec{n})}{\xi(\vec{n})} \tag{8.15b}$$

and the metric is given by

$$g_{ij}^{(2)} = \mathcal{A} \frac{\overline{c}_l c_r}{\pi |c|^4} \left(\delta_{ik} \frac{\overline{c}_s c_t}{|c|^2} S_{lrst} - S_{lrik} - S_{lkir} \right) \left(\delta_{jk} - \frac{\overline{c}_j c_k}{|c|^2} \right) \tag{8.16}$$

8.1.4 Higher order computations

Although it is possible to obtain closed analytical expressions for higher orders in the Bradlow parameter expansion, it is more convenient to switch to numerical computation of higher orders.

Since the Fourier modes $h(\vec{n})$ decrease very fast with the mode number $\vec{n}$, we can obtain a very good (i.e machine precision) approximation to the metric by cutting the infinite sums of the convolutions to a handful of mode numbers. In practice the situation here is even better than the one that we had to study the solutions, since here the metric is given *only* by the zero Fourier mode of a convolution.

The code `metric.out`, documented in the appendix (see E.3), performs these operations. All the numerical data that we will show later has been computed with the help of this code.

8.2 Near the Bradlow limit

Here we will address the problem of computing the dynamics for small Torus, near the Bradlow limit ($\epsilon \to 0$), when with a couple of orders we get a very good approximation to the full metric.

We will compute the metric in the (θ, ψ) basis previously defined (equation (7.40)). Since the metric takes the form of equation (7.41), we will compute the $\epsilon-$power expansion of the function $f(\theta, \psi)$. As we have seen this function can be expanded in the following way:

$$f^{(n)}(\theta, \psi) = \frac{\mathcal{A}}{\pi} \sum_{j,k=0}^{j+2k \leq n-1} A_{jk}^{(n)} \cos^{2j}\theta \sin^{4k}\theta \cos^{2k} 2\psi \tag{8.17}$$

Using the code `metric.out` (section E.3) it is relatively easy to obtain this coefficients numerically. The first five orders are given up to machine precision in the table (8.1).

We will now analyse the two-vortex dynamics that follows from our fifth order metric. This is governed by the Hamiltonian

$$H = \frac{1}{f(\theta, \varphi)} \left(p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta} \right) \tag{8.18}$$

	$\mathbf{O(3)}$		$\mathbf{O(2)}$
$A^{(3)}_{00}$	$1.57\ 24\ 57\ 43\ 63\ 76\ 56(1) \times 10^{-1}$	$A^{(2)}_{00}$	$2.09\ 85\ 65\ 63\ 87\ 72\ 34\ 78\ 07 \times 10^{-1}$
$A^{(3)}_{10}$	$-7.35\ 97\ 58\ 95\ 32\ 71\ 25 \times 10^{-1}$	$A^{(2)}_{10}$	$-6.29\ 56\ 96\ 91\ 63\ 17\ 0439\ 78 \times 10^{-1}$
$A^{(3)}_{20}$	$4.40\ 39\ 77\ 74\ 02\ 36\ 00 \times 10^{-1}$		$\mathbf{O(5)}$
	$\mathbf{O(4)}$	$A^{(5)}_{00}$	$3.24\ 86\ 80\ 02\ 33\ 12\ 76 \times 10^{-1}$
$A^{(4)}_{00}$	$2.56\ 44\ 31\ 24\ 38\ 37\ 31\ 64(1) \times 10^{-1}$	$A^{(5)}_{10}$	$-1.59\ 28\ 19\ 34\ 17\ 36\ 19\ 6$
$A^{(4)}_{10}$	$-1.09\ 63\ 43\ 50\ 68\ 01\ 26(2)$	$A^{(5)}_{20}$	$2.35\ 30\ 76\ 79\ 79\ 16\ 14\ 0$
$A^{(4)}_{20}$	$1.12\ 62\ 17\ 59\ 37\ 90\ 85(1)$	$A^{(5)}_{30}$	$-1.33\ 17\ 87\ 41\ 82\ 85\ 50\ 3$
$A^{(4)}_{30}$	$-3.23\ 47\ 12\ 72\ 58\ 54\ 03(4) \times 10^{-1}$	$A^{(5)}_{40}$	$2.44\ 37\ 72\ 51\ 19\ 98\ 28 \times 10^{-1}$
$A^{(4)}_{01}$	$-2.62\ 60\ 73\ 46\ 53\ 83\ 54(1) \times 10^{-1}$	$A^{(5)}_{01}$	$-4.46\ 53\ 65\ 89\ 63\ 78\ 00 \times 10^{-1}$
		$A^{(5)}_{11}$	$4.62\ 92\ 29\ 53\ 30\ 50\ 51 \times 10^{-1}$

Table 8.1: Numerical values of the coefficients that determines the metric up to fifth order.

The equations of motion are

$$\dot\theta = \frac{2p_\theta}{f} \tag{8.19a}$$

$$\dot\varphi = \frac{2p_\varphi}{f\sin^2\theta} \tag{8.19b}$$

$$\dot p_\theta = \frac{1}{f}\left(E\frac{\partial f}{\partial\theta} + 2p_\varphi^2\frac{\cos\theta}{\sin^3\theta}\right) \tag{8.19c}$$

$$\dot p_\varphi = \frac{E}{f}\frac{\partial\tilde f}{\partial\varphi} \tag{8.19d}$$

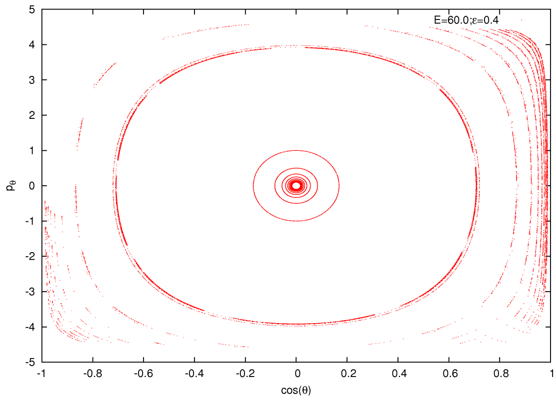
Where E is the energy of the system, that is conserved. These equations can be easily integrated numerically to obtain the trajectories in vortex moduli space.

Notice, that up to third order, the metric is independent of φ. Hence, p_φ is a conserved quantity, and the problem is integrable. In this case, the equation for the trajectory in the moduli space is given by

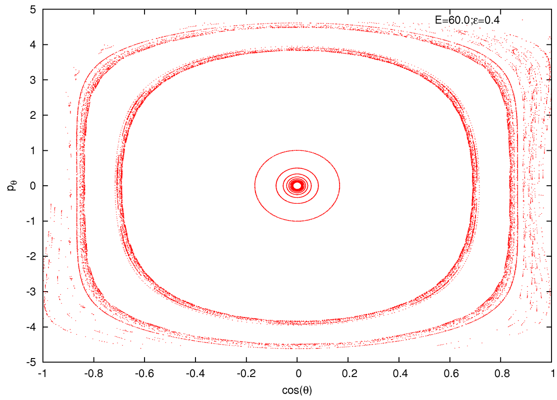
$$\varphi_f = \varphi_0 + \int_{\theta_0}^{\theta_f} \frac{d\theta}{\sin^2\theta\sqrt{\frac{f(\theta)}{b^2} - \frac{1}{\sin^2\theta}}} \tag{8.20}$$

where $b = \frac{p_\varphi}{\sqrt{E}}$.

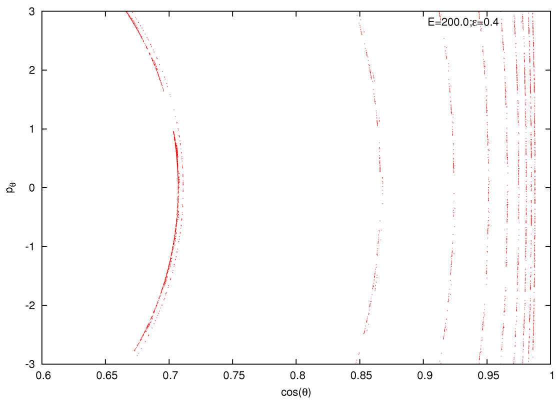
An interesting mathematical question is whether the two vortex dynamics on the torus is integrable for all values of ϵ. As we have seen, this is the case up to third order in the expansion. In the next subsection we will mention that it is also the case in the $\epsilon \longrightarrow 1$ limit, since then we recover rotational invariance and angular momentum becomes a conserved quantity. Our metric calculations could be used to give a numerical and/or analytical answer to this question. We will not attempt to study that point here. Instead we will address the much simpler problem of obtaining and describing the Poincare maps obtained from the third order and fifth order metric. Results are displayed in Fig. 8.1 for two values of ϵ. We see signs of how some invariant-tori seem to be destroyed, pointing to the non-integrability of the metric truncated to fifth order. This is not surprising and does not conflict with the conjectured integrability for all values of ϵ. Our results, nonetheless, can be considered as a first step, which might help in pointing to the relevant region of phase space where one should look.



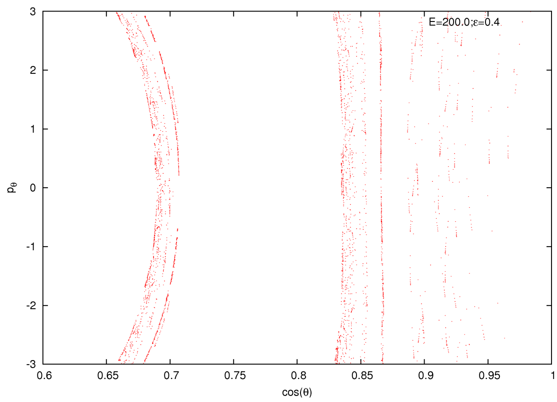
(a) Poincare map for $\epsilon = 0.4$ and $E = 60.0$ for the first 3 orders.



(b) Poincare map for $\epsilon = 0.4$ and $E = 60.0$ for the first 5 orders.



(c) Detail of the Poincare map for $\epsilon = 0.4$ and $E = 200.0$ for the first 3 orders.



(d) Detail of the Poincare map for $\epsilon = 0.4$ and $E = 200.0$ for the first 5 orders.

Figure 8.1: Examples of Poincare maps for the dynamics near the Bradlow limit.

8.3 Dynamics of vortices in the plane (the $\epsilon \to 1$ limit).

The dynamics of the Abelian Higgs model in $\mathbb{R}^2$ is a well studied problem, both analytically [79, 75] and numerically [64, 75]. Following the notation of Ref. [75] we might write

$$ds^2 = 8F^2(2|u|)\, du d\overline{u} \tag{8.21}$$

Once the function $F(2|u|)$ is obtained, the dynamics in the geodesic approximation is completely determined. The form of the function $F(2|u|)$ has been deduced analytically for the special case of asymptotically separated vortices, as we have seen, and has also been studied numerically several times[79, 75]. Here we will use our expansion method to obtain a precise determination of $F(2|u|)$ for all values of $|u|$.

We will naturally assume that, if the torus is large enough compared both with the typical size of vortices, and with the distance between them, the *dynamics* will be accurately described by the one in $\mathbb{R}^2$. This means that, at least formally, we can calculate the desired function $F(2|u|)$ by taking the $\epsilon \longrightarrow 1$ limit while keeping u fixed. We will use the zeros of the Higgs field as coordinates in the moduli space. The expression for the metric in this coordinate system (equation (7.45)) involves $f(z,\overline{z})$ times a known function arising from the change of coordinates. We will first study the behaviour of the latter factor in the appropriate limit ($\epsilon \longrightarrow 1\,,\ u$ fixed). Restricting to the $l_1 = l_2 = l$ case and recalling that $w = u\sqrt{2\pi(1-\epsilon)}$ we get:

$$\left(\frac{4\pi}{l}\right)^2 |\vartheta_4(0)|^4 \frac{|\vartheta_1(w)|^2|\vartheta_4(w)|^2}{(|\vartheta_2(w)|^2+|\vartheta_3(w)|^2)^2} \stackrel{\epsilon\to 1}{\longrightarrow} \frac{16\pi^2 Z_2^2}{(1+Z_0^2)^2}(1-\epsilon)^2|u|^2 \tag{8.22}$$

where[2] $Z_2 = -\frac{1}{2}\frac{\vartheta_3(0)\vartheta_4^4(0)}{\vartheta_2(0)}$, and $Z_0 = -\vartheta_3(0)/\vartheta_2(0)$.

In summary, the function $F^2(2|u|)$ can be recovered as

$$F^2(2|u|) = \lim_{\epsilon\longrightarrow 1} (1-\epsilon)^2 \tilde{f}_w(u\sqrt{2\pi(1-\epsilon)}, \overline{u}\sqrt{2\pi(1-\epsilon)}, \epsilon)\ \frac{2\pi^2 Z_2^2}{(1+Z_0^2)^2}|u|^2 \tag{8.23}$$

In practice, to obtain an approximation to $F^2(2|u|)$ from our expansion we proceeded as follows. Using the code `metric.out` we computed the coefficients in the expansion in ϵ of the metric up to 40th order, for a regular rectangular array of points in the complex plane of the variable w. The range of values is that of a complex torus. Due to symmetries (which we checked) it is enough to take values in one quadrant. We considered a grid of 40×40 points. Getting to order 40 is feasible and takes around 50 hours in a standard PC. The truncation in the number of Fourier modes is tuned so that the precision attained in the function $f(z,\overline{z})$ is of 14-16 decimal places (in practise this means that you need around $16-20$ Fourier modes). Due to the fast convergence of the Fourier coefficients, this procedure is not very costly.

Once $f(z,\overline{z})$ is determined at the grid points, in order to extract the values of $F^2(2|u|)$ we proceeded as follows. For a given value of $|u|$ we take many different values of w from the grid and for each we compute the corresponding value of ϵ. For that value we sum the series up to 40th order and multiply by the function associated to the change of variables (equation (7.46)). Finally, for each value of $|u|$ we obtain we obtain a collection of numbers for different values of ϵ and for different phases of u. A typical case is displayed in the figure (8.2), where we considered points having $|u| = 1$ and situated along the diagonal $\mathrm{Re}(u) = \mathrm{Im}(u)$. The value of $F^2(2|u|)$ is the limit for ϵ approaching 1. Unfortunately, the larger values of ϵ are more seriously affected by the truncation of the series to 40 orders. This is exemplified by the errors attached to the points of the figure. These are obtained by subtracting the result obtained for 35 orders to that of 40 orders. To obtain a more precise determination of the value of $F(2|u|)$ we made two corrections. First of all, we partially corrected for the truncation errors by estimating the contribution of the terms from the 41st on. The precise way in which this is done will be explained later. This additional contribution does not affect significantly points for which the error obtained by comparing the 35th and 40th order is tiny. On the opposite extreme the contribution becomes unreliable when this error becomes very large. This occurs

[2]For the rectangular symmetric case $l_1 = l_2$, the numerical values of Z_0 and Z_2 are

$$\begin{aligned} Z_0 &= -1-\sqrt{2} \\ Z_2 &= -1.18\ 91\ 73\ 93\ 79\ 71\ 08\ 15 \end{aligned}$$

typically for values of ϵ exceeding 0.95 (areas which are more than 20 times the critical area). We will therefore omit those points from our analysis.

To extract a good determination of $F^2(x)$ from our data we face the problem of extrapolating to $\epsilon = 1$. This implies relating the results for a large torus with that of the plane. The torus case can be considered a solution of the plane with an infinite number of vortices – two per $l \times l$ cell. As the torus gets larger the replica vortices are increasingly far away, so that it is reasonable to use the analytic result for the case of a large number of asymptotically separated vortices to describe the approach. Using the result of the form of the metric for asymptotically separated vortices described in the section 6.4 we expect an additional contribution to the metric proportional to $\sum_r K_0(D_r)$, where K_0 is the modified Bessel function of the second kind and D_r is the distance to the replica vortex r. In the limit of large torus sizes and fixed $|u|$, this predicts a contribution proportional to $K_0(l)$. On the basis of the asymptotic behaviour of Bessel functions we decided to fit our data for each value of u to a formula of the form

$$A - B(1-\epsilon)^{\frac{1}{4}} e^{-\sqrt{8\pi/(1-\epsilon)}} \tag{8.24}$$

For our $|u| = 1$ example, the result is displayed by the solid line in the figure (8.2), and the fitted parameters are $B = 145.0(5)$ and $A = F^2(2) = 0.8791(4)$. The error reflects systematics due to changing the range of the fit as well as the relative weighting of the points. For other phases of u we get compatible results. For example, for purely real or imaginary values of u we obtain $F^2(2) = 0.8800(6)$, consistent with rotational invariance.

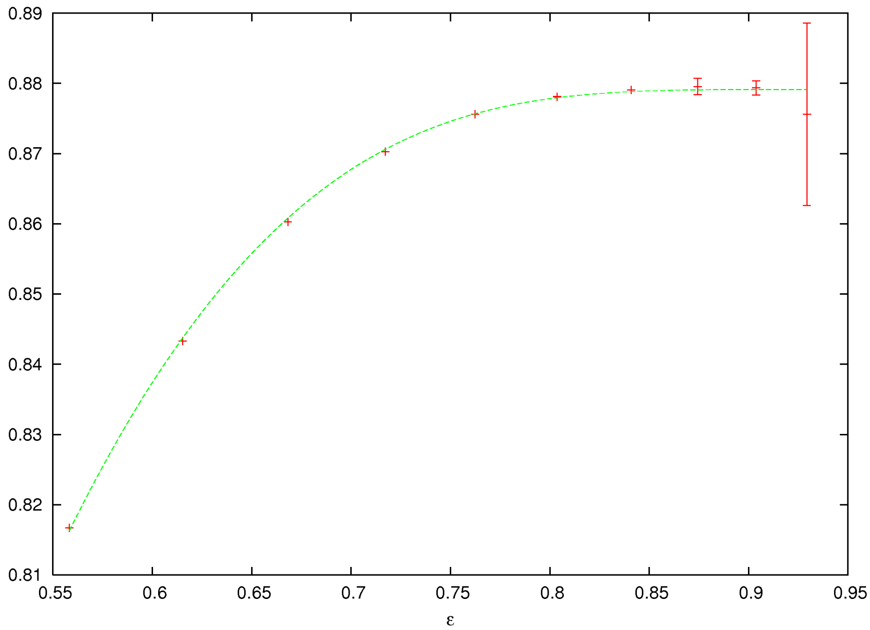


Figure 8.2: The points are the values of the left hand side of Eq. 8.23 for $|u| = 1.0$ as a function of ϵ, computed from the first 40 orders of our expansion. Error bars represent the size of the last 5 terms of the expansion. The solid line is a fit with equation (8.24).

Proceeding in the same way for several values of $|u|$ we arrive at the points displayed in the figure (8.3). The errors are smaller than the size of the points in the figure. In all the range up to $|u| \geq 2$ the error is smaller than 6×10^{-4} and for $|u| > 0.5$ the relative error is smaller than 1 part in 10^3.

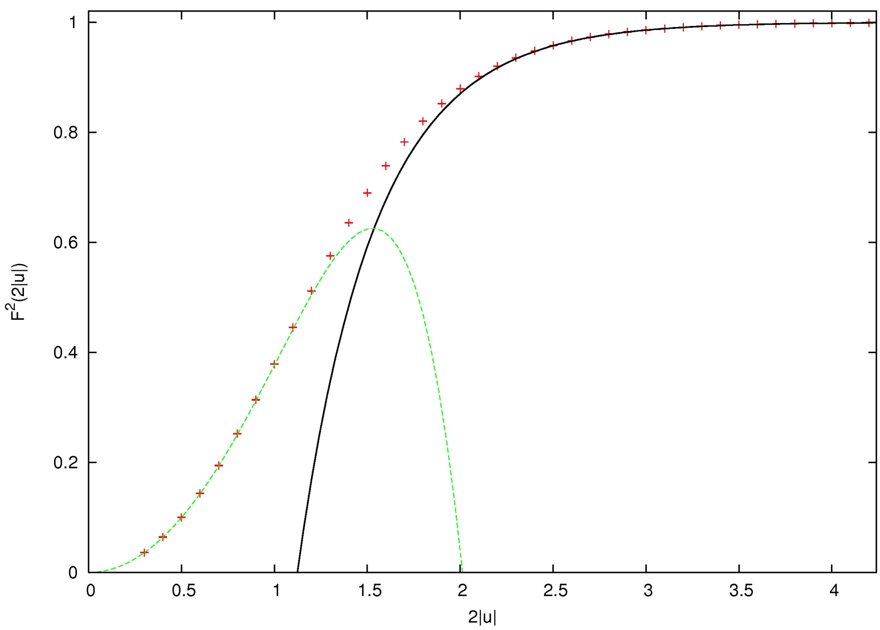


Figure 8.3: The metric computed using 40 orders of the Bradlow parameter expansion (red points), versus the $|u| \to 0$ approximation Eq. 8.29 (green dashed line) and the asymptotic form of the metric Eq. 8.30 (black solid line).

It is quite interesting to analyse the implications of the scaling limit for the coefficients of the ϵ expansion. For that purpose it is convenient to write $\epsilon = e^{-\delta}$ and replace the limit by $\delta \to 0$. The condition that at large volumes the metric tends to that of the plane implies:

$$F^2(2|u|) = \lim_{\delta \to 0} \delta^2 \frac{2\pi^2 Z_2^2}{(1+Z_0^2)^2} |u|^2 \sum_{n=1}^{\infty} f^{(n)}(u\sqrt{2\pi\delta}, \overline{u}\sqrt{2\pi\delta}) e^{-n\delta} \tag{8.25}$$

It is clear that as δ tends to zero, more terms in the series become relevant. It is tempting to assume that the infinite sum tends to an integral over the variable $y = n\delta$. This would imply the following behaviour of the coefficients:

$$\lim_{n \longrightarrow \infty} f^{(n)} \left(u\sqrt{\frac{2\pi y}{n}}, \overline{u}\sqrt{\frac{2\pi y}{n}} \right) \frac{\delta}{8} = h(|u|\sqrt{y}) \tag{8.26}$$

In this case the limiting function $h(|u|\sqrt{y})$ would be related to $F^2(2|u|)$ as follows:

$$F^2(2|u|) = \frac{16\pi^2 Z_2^2}{(1+Z_0^2)^2} |u|^2 \int_0^{\infty} e^{-y} h(|u|\sqrt{y})\, dy \tag{8.27}$$

which is essentially a Laplace transform.

With our numerical results (computed with the code `metric.out`) for $f^{(n)}$ up to $n = 40$ and the set of values of w spanning the aforementioned 40×40 grid, we have analysed the validity of equation (8.26). For that purpose we plot in the figure (8.4) the coefficients satisfying $35 \leq n \leq 40$ and all values of u in the range. The smoothness and thinness of the resulting curve, approximating $h(x)$, is a test of our scaling hypothesis. Errors, signalled by the spread of the values, grow with x. For example, including all coefficients $n > 20$ would provide little changes in the thickness of the curve up to $x \approx 5$. Beyond, the thickness becomes a sizable fraction of the value.

The function $h(x)$ for small x, as determined from our results, fits nicely to a behaviour of the form

$$h(x) = a - bx^4 + O(x^5) \tag{8.28}$$

where $a = 0.335(1)$ and $b = 0.16(1)$. This expansion implies that for small values of the argument the metric function $F^2(2|u|)$ behaves as:

$$F^2(2|u|) = \frac{16\pi^2 Z_2^2}{(1+Z_0^2)^2}\left(a|u|^2 - 2b|u|^6\right) + O(|u|^7) \tag{8.29}$$

The function on the left-hand side of the previous formula is displayed in the figure (8.3). It matches nicely to the behaviour of the previously determined points (in red). For large values of $|u|$ the data also matches with the prediction coming from the asymptotic behaviour at large distances (Ref. [76]) given by

$$F^2(2|u|) \stackrel{|u|\to\infty}{\longrightarrow} 1 - 8\sqrt{2}K_0(4|u|) \tag{8.30}$$

where $K_0(x)$ is the modified Bessel function of the second kind.

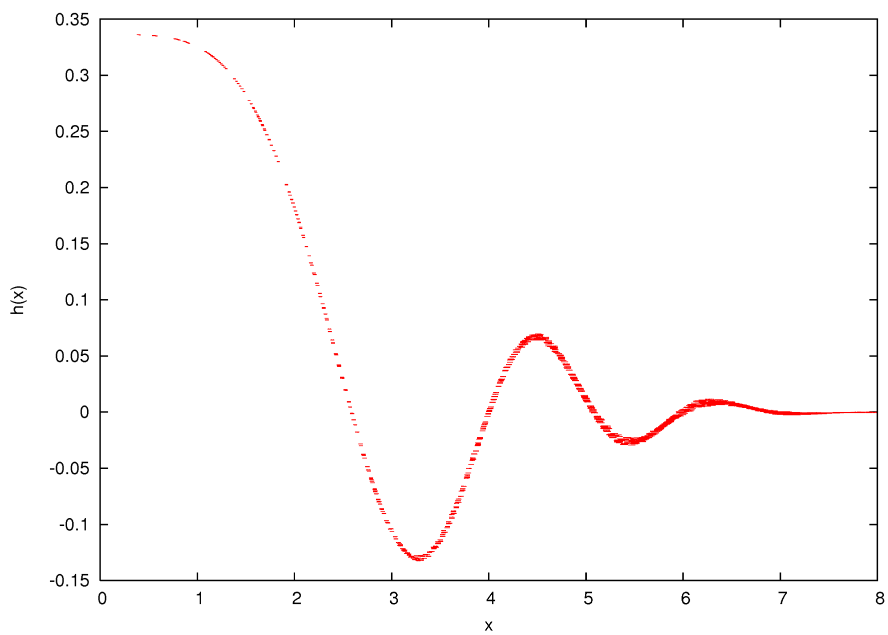


Figure 8.4: We plot the left-hand side of Eq. 8.26 for n in the range 35-40, and fixed $x = |u|\sqrt{y}$.

To conclude we point out that our knowledge of $h(x)$ enables one to estimate the error introduced by the truncation in the number of terms in the ϵ expansion. For that one has simply to use the expression given in Eq. 8.27 with the integral restricted to $y > -n\log(\epsilon)$, where n is the maximum order computed. We have tested this estimation with our data points and $n = 40$. As mentioned previously, it improves the results for not too large values of ϵ.

Scattering of vortices on the plane

Once the function $F(2|u|)$ has been obtained, it is trivial to compute the dynamics of the vortices. The Lagrangian, when written in polar coordinates $2u = re^{\imath\theta}$, is given by:

$$T = \pi F^2(r)\left[\dot{r}^2 + r^2\dot{\theta}^2\right] \tag{8.31}$$

Since the energy is conserved, and we have rotational invariance (p_θ is also conserved), the system is integrable, and we have

$$\theta_f = \pi - \int_{r_f}^{\infty} \frac{dr}{r\sqrt{\frac{r^2}{b^2}F^2(2r) - 1}} \tag{8.32}$$

where $b^2 = \frac{\pi p_\theta^2}{4T}$ is the *impact parameter*. In particular, if we call r_m the minimum distance between one vortex and the centre of mass during the trajectory, the scattering angle is given by

$$\theta_{sc} = \pi - 2\int_{r_m}^{\infty} \frac{dr}{r\sqrt{\frac{r^2}{b^2}F^2(r) - 1}} \tag{8.33}$$

Using our numerical data of the figure (8.3) we can calculate these integrals. For values of $r > 5$ we will use the asymptotic form of the metric, and for $r < 0.5$ we can use our approximation equation (8.29). For the intermediate values we will use the cubic spline interpolating polynomial. In the figure (8.5) we have some examples of trajectories for different values of the impact parameter.

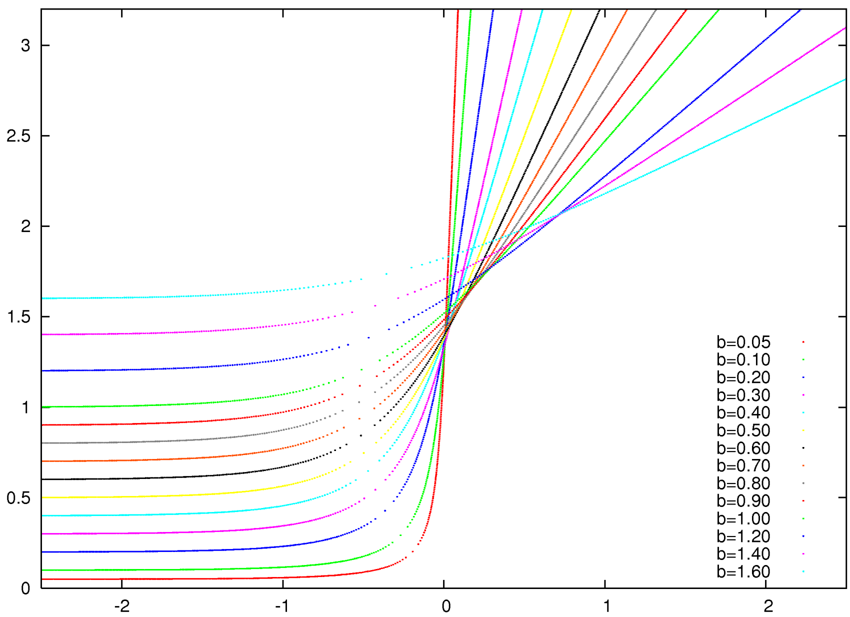


Figure 8.5: Scattering trajectories of two vortices.

We can also plot the scattering angle versus the impact parameter. The results are shown in the figure (8.6). Errors were estimated in the following way. We generate a random set of values of $F(2|u|)$ assuming a Gaussian distribution around the central value and standard deviation fixed by the error. For each realization we compute the cubic spline interpolation polynomial and using it and the small and large approximation functions we compute the scattering angle for each impact parameter. The final errors are estimated on the basis of 1000 realizations. They are of the order of 1 part in 10^4 or smaller.

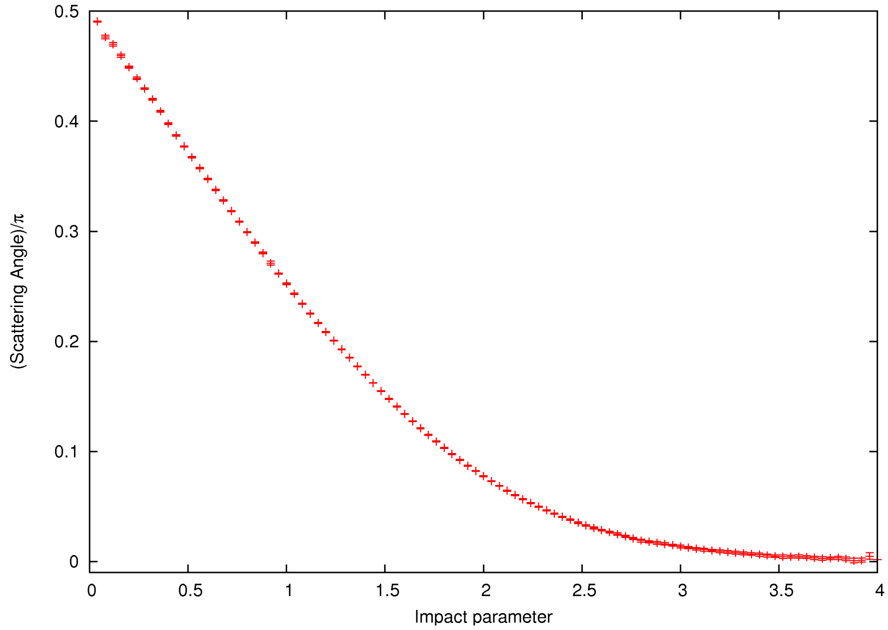


Figure 8.6: Scattering Angle vs. impact parameter for the two vortices in the plane.

8.4 The expansion of the metric for semi local strings

As we have seen in section 7.4, the expression for the metric of the moduli space of semilocal strings living in the torus is

$$g_{ij}^{(a,b)} = \frac{\epsilon}{\pi} \int d^2x \, e^{-2h} \left(\chi_i \delta_{ab} - c_k^{(a)} \chi_k \mathcal{H}_{i(b)} \right) \overline{\chi}_j \tag{8.34}$$

where h is the doubly periodic function solution of the vortex equation ($M_{ss'} = c_s^{(a)} \overline{c}_{s'}^{(a)}$)

$$\overline{\partial} \partial h = \frac{\epsilon}{2} \left(1 - e^{-2h} M_{ss'} \overline{\chi}_s \chi_{s'} \right) \tag{8.35}$$

and $\mathcal{H}_{i(b)}$ obeys the equation

$$\hat{O} \mathcal{H}_{j(b)} = e^{-2h} \overline{c}_i^{(b)} \overline{\chi}_i \chi_j \tag{8.36}$$

The positive definite operator $\hat{O}$ is

$$\hat{O} = -\partial \overline{\partial} + \sum_a \left| \phi^{(a)} \right|^2 \tag{8.37}$$

The idea of the Bradlow parameter expansion is essentially the same here: if we have both h and $\mathcal{H}_{i(a)}$ as a power series in the Bradlow parameter ϵ, equation (8.34) will also induce a power expansion for the metric

$$g_{ij}^{(a,b)} = \sum_n g_{ij}^{(a,b)(n)} \epsilon^n \tag{8.38}$$

To obtain the functions h and $\mathcal{H}_{i(a)}$ one needs to solve the equations

$$\begin{aligned} \overline{\partial}\partial h &= \frac{\epsilon}{2}\left(1 - e^{-2h} M_{ss'} \overline{\chi}_s \chi_{s'}\right) && (8.39a) \\ \overline{\partial}\partial \mathcal{H}_{j(b)} &= e^{-2h}\left(\overline{c}_i^{(a)} c_k^{(a)} \mathcal{H}_{j(b)} \chi_k - \overline{c}_i^{(b)} \chi_j\right) \overline{\chi}_i && (8.39b) \end{aligned}$$

The constant part of these functions should be obtained by integrating both sides of the equation in the torus

$$\begin{aligned} \frac{1}{\mathcal{A}} \int d^2x\, e^{-2h} M_{ss'} \overline{\chi}_s \chi_{s'} &= 1 && (8.40a) \\ \int d^2x\, e^{-2h} \overline{c}_i^{(a)} \overline{\chi}_i \mathcal{H}_{j(b)} \chi_k c_k^{(a)} &= \int d^2x\, e^{-2h} \overline{c}_i^{(b)} \overline{\chi}_i \chi_j && (8.40b) \end{aligned}$$

The procedure to solve these equations is exactly the same as in the case of the study of the metric for non extended vortices. we will both expand h and $\mathcal{H}_{i(a)}$ in a Fourier series

$$\begin{aligned} h &= \sum_{\vec{n}} h(\vec{n}) e^{2\pi\imath\left(n_1 \frac{x_1}{l_1} + n_2 \frac{x_2}{l_2}\right)} && (8.41a) \\ \mathcal{H}_{i(a)} &= \sum_{\vec{n}} \mathcal{H}_{i(a)}(\vec{n}) e^{2\pi\imath\left(n_1 \frac{x_1}{l_1} + n_2 \frac{x_2}{l_2}\right)} && (8.41b) \end{aligned}$$

and at the same time we will expand each Fourier mode as a power series in the Bradlow parameter

$$\begin{aligned} h(\vec{n}) &= \sum_k h^{(k)}(\vec{n}) \epsilon^k && (8.42a) \\ \mathcal{H}_{i(a)}(\vec{n}) &= \sum_n \mathcal{H}_{i(a)}^{(k)}(\vec{n}) \epsilon^k && (8.42b) \end{aligned}$$

Now we have to solve these equations order by order. The procedure is completely analogous to the case of the non extended abelian Higgs model, so we will not repeat here the details. In the end of the day, we arrive to an expression for the metric

$$g_{ij}^{(a,b)(n)} = \frac{\mathcal{A}}{\pi} \left\{ \sum_{\vec{k}} P^{(n)}(\vec{k}) L^{ij}(\vec{k}) \delta_{ab} - c_k^{(a)} \sum_{l=0}^{n} \sum_{\vec{k},\vec{k}'} P^{(l)}(\vec{k}) L^{kj}(\vec{k}') \mathcal{H}_{i(b)}^{(n-l)}(\vec{k}' - \vec{k}) \right\} \tag{8.43}$$

8.4.1 Zero order computation

As in the previous case, the metric is directly proportional to ϵ, so the metric is zero

$$g_{ij}^{(a,b)(0)} = 0 \tag{8.44}$$

8.4.2 First order computation

To compute the metric up to first order, it is enough to know $h^{(0)}(\vec{n})$ and $\mathcal{H}^{(0)}_{i(a)}(\vec{n})$,, and it is easy to see that only the constant Fourier term is different from zero

$$h^{(0)}(0) = \ln \|c\| \tag{8.45a}$$

$$\mathcal{H}^{(0)}_{i(a)}(0) = \frac{\overline{c}_i^{(a)}}{\|c\|^2} \tag{8.45b}$$

where $\|c\|^2 = \overline{c}_i^{(a)} c_i^{(a)}$. This gives the following result for the metric

$$g_{ij}^{(a,b)(0)} = \frac{\mathcal{A}}{\pi \|c\|^2} \left(\delta_{ij}\delta_{ab} - \frac{c_i^{(a)} \overline{c}_j^{(b)}}{\|c\|^2} \right) \tag{8.46}$$

8.4.3 Higher order computations

As in the case of the non extended abelian Higgs model, one have to switch to numerical computations to perform high order computations. The case is so similar, that the same considerations that we said there are valid here.

Nine

Eigenvalues of the Dirac operator

In this chapter we will study the eigenvalues and eigenfunctions of the Dirac operator in the presence of vortex solutions. This is part of a work in progress [80].

9.1 Basics

The Dirac operator is given by

$$\not{D} = \gamma^i D_i = \gamma^i(\partial_i - \imath A_i) \tag{9.1}$$

where $A_i(x)$ is the gauge potential of a vortex solution in the Torus. Using the representation for the gamma matrices given in the appendix A

$$\gamma^1 = \imath\tau_2; \quad \gamma^2 = -\imath\tau_1 \tag{9.2}$$

we have

$$\not{D} = \begin{pmatrix} 0 & D_+ \\ -D_- & 0 \end{pmatrix} \tag{9.3}$$

where the operators $D_\pm$ are given by

$$\begin{aligned} D_+ &= D_1 - \imath D_2 && (9.4a) \\ D_- &= D_1 + \imath D_2 && (9.4b) \end{aligned}$$

We are interested in the eigenvalues and eigenfunctions of the Dirac operator. The two component spinors ψ

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \tag{9.5}$$

obey the usual quasi periodicity conditions studied in detail in appendix C

$$\begin{aligned} \psi(x + \vec{l}_1) &= e^{\imath\pi q \frac{x_2}{l_2}}\psi(x) && (9.6a) \\ \psi(x + \vec{l}_2) &= e^{-\imath\pi q \frac{x_1}{l_1}}\psi(x) && (9.6b) \end{aligned}$$

The eigenvalue equation reads

$$\not{D}\psi = \lambda\psi \tag{9.7}$$

The strategy to solve this eigenvalue equation is the following: using the techniques of chapter 5 we can obtain the gauge potential for the vortex solutions as a power expansion in the Bradlow parameter, then we can obtain also the Dirac operator $\not{D}$ as a power expansion in the Bradlow parameter

$$\epsilon = 1 - 2f = 1 - \frac{4\pi q}{l_1 l_2} \tag{9.8}$$

that is given by

$$\not{D} = \sum_{n=0}^{\infty} \not{D}^{(n)} \epsilon^n = \begin{pmatrix} 0 & \sum_{n=0}^{\infty} D_+^{(n)} \epsilon^n \\ -\sum_{n=0}^{\infty} D_-^{(n)} & 0 \end{pmatrix} \tag{9.9}$$

the zero order matrix of this expansion

$$\not{D}^{(0)} = \begin{pmatrix} 0 & D_+^{(0)} \\ -D_-^{(0)} & 0 \end{pmatrix} \tag{9.10}$$

has a known spectrum, since it correspond to the problem of the Landau energy levels of a particle in a constant magnetic field. From our point of view, this case can be solved easily because the operators $D_\pm^{(0)}$ can be interpreted as creation and annihilation operators acting in the Hilbert space of quasi periodic functions studied in the appendix C. Once we know the spectrum of the operator $\not{D}^{(0)}$, the techniques developed in the appendix D can be used to obtain the spectrum of the operator $\not{D}$ as a power expansion in the Bradlow parameter ϵ.

To compute the explicit solution to zero order in ϵ, we first write the eigenvalue equation

$$\begin{pmatrix} D_+^{(0)} \psi_- \\ -D_-^{(0)} \psi_+ \end{pmatrix} = \lambda \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \tag{9.11}$$

since, as we said, the operators $D_\pm^{(0)}$ can be written in terms of creation and annihilation operators

$$D_+^{(0)} = \imath\sqrt{2f} a^+ \tag{9.12a}$$
$$D_-^{(0)} = \imath\sqrt{2f} a^- \tag{9.12b}$$

it is easy to see that the eigenfunctions are given by

$$\psi_{nsp}^{(0)} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mid n, s\rangle \\ \imath p \mid n-1, s\rangle \end{pmatrix} \tag{9.13}$$

where the label $n \in \mathbb{Z}^+$ run over the positive integers (the case $n = 0$ correspond to the zero modes, and will be treated in detail later), $s \in \mathbb{Z}_q$ is an integer modulo q, the label $p = \pm 1$ is a sign, and the functions $\mid n, s\rangle$ are elements of a basis of the Hilbert space of quasi periodic functions whose explicit form is given in the definition C.1.1 (page 161). The corresponding eigenvalues are

$$\lambda_{np}^{(0)} = -p\sqrt{2fn} \tag{9.14}$$

so we clearly see that all the eigenvalues are q times degenerated, but the zero modes, that deserve special attention.

9.1.1 The zero mode

The eigenvalue equations for the zero mode reads

$$\begin{pmatrix} D_+\psi_- \\ -D_-\psi_+ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{9.15}$$

so we have to solve the pair of equations

$$D_+\psi_- = 0 \tag{9.16a}$$
$$D_-\psi_+ = 0 \tag{9.16b}$$

We have faced these equations before, in section 4.4. There we found the solutions to these equations, but here we will repeat the argument. As usual, using complex notation (see section A.4), and the Hodge decomposition theorem, we will write the gauge connection as

$$A = -\imath\partial\overline{H} + v - \imath f\overline{z} \tag{9.17}$$

where H is a doubly periodic complex function and v is a complex constant. Writing

$$\psi_+ = e^{-H}\chi_+^{(v)} \tag{9.18a}$$
$$\psi_- = e^{\overline{H}}\chi_-^{(v)} \tag{9.18b}$$

the fields $\chi_\pm$ obey the equation

$$D_+^{(0)}\chi_- = 0 \tag{9.19a}$$
$$D_-^{(0)}\chi_+ = 0 \tag{9.19b}$$

and interpreting these equations in terms of creation and annihilation operators, we obtain the solutions to these equations

$$\psi_+ = e^{-H} \mid 0, s\rangle \tag{9.20a}$$
$$\psi_- = 0 \tag{9.20b}$$

where $s \in \mathbb{Z}_q$ is an integer modulo q. As we expected from the index theorem, we have q linearly independent zero modes. This result is exact to all orders in ϵ.

9.2 The Bradlow parameter expansion for the eigenvalues

The idea now is to use the techniques of the appendix C to compute the spectrum as a power series in the parameter ϵ

$$\lambda_{nsp} = \sum_{n=0}^{\infty} \lambda_{nsp}^{(n)}\epsilon^n \tag{9.21}$$

we know that the zero order is given by

$$\lambda_{nsp}^{(0)} = -p\sqrt{2fn} \tag{9.22}$$

with eigenfunctions

$$\psi_{nsp}^{(0)} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mid n, s\rangle \\ \imath p \mid n-1, s\rangle \end{pmatrix} \tag{9.23}$$

As usual we will use complex notation (see appendix A.4) and the Hodge decomposition theorem to write the gauge connection

$$A = -\imath\partial\overline{H} + v - \imath f\overline{z} \tag{9.24}$$

where H is a doubly periodic function that can be obtained as a power expansion in ϵ (see chapter 5)

$$H = \sum_{n=1}^{\infty} H^{(n)}\epsilon^n \tag{9.25}$$

The general form of $\not{D}^{(k)}$ for $k > 0$ is given by

$$\not{D}^{(k)} = -\begin{pmatrix} 0 & \partial\overline{H}^{(k)} \\ \overline{\partial}H^{(k)} & 0 \end{pmatrix} \tag{9.26}$$

The case $q = 1$ is very different from the case $q > 1$ because in the first case the eigenstates are non degenerated, but in the second case they are degenerated. Now we will show some explicit examples.

9.2.1 $q = 1$ case

In this case, the label s of both the eigenvalues and the eigenfunctions to zero order in ϵ is useless (has always the same value), so the eigenfunctions will be labelled simply $\psi_{np}^{(0)}$, and the eigenvalues $\lambda_{np}^{(0)}$. Before computing an explicit example, we will define some integrals that will be useful in the future

$$I_{n,n'}^{(k)} = \langle n, 0 \mid \partial\overline{H}^{(k)} \mid n', 0\rangle \tag{9.27a}$$

$$\overline{I}_{n,n'}^{(k)} = \langle n, 0 \mid \overline{\partial}H^{(k)} \mid n', 0\rangle \tag{9.27b}$$

$$\tag{9.27c}$$

The scalar product is defined in appendix C, equation C.2. The symbols $I_{n,n'}^{(k)}$ obey the property

$$I_{n,n'}^{(k)} = (\overline{I}_{n',n}^{(k)})^* \tag{9.28}$$

We can write the general term that will appear in the expansion for the eigenvalues in terms of these integrals

$$\langle\psi_{np}^{(0)} \mid \not{D}^{(k)} \mid \psi_{n'p'}^{(0)}\rangle = \frac{-\imath p'}{2}\left[I_{n,n'-1}^{(k)} - \frac{p}{p'}\overline{I}_{n-1,n'}^{(k)}\right] \tag{9.29}$$

First order correction

To this order the correction is given by

$$\lambda_{np}^{(1)} = \langle\psi_{np}^{(0)} \mid \not{D}^{(1)} \mid \psi_{np}^{(0)}\rangle \tag{9.30}$$

And using the results of the previous section, this is simply given by

$$\lambda_{np}^{(1)} = p\mathrm{Im}\left(I_{n,n-1}^{(1)}\right) \tag{9.31}$$

The integral $I^{(1)}_{n,n-1}$ can be computed exactly noting that the Fourier modes of $H^{(1)}$ can be computed (see chapter 5, in special section 5.1.1)

$$\mathcal{F}\left[\partial\overline{H}^{(1)}\right] = (-)^{k_1 k_2}\frac{2\pi\imath}{\xi}\sqrt{\frac{\tau(1-\epsilon)}{4\pi q}}\left(k_1 - \imath\frac{k_2}{\tau}\right)e^{-\xi/2} \tag{9.32}$$

where

$$\xi = \pi\tau\left(k_1^2 + \frac{k_2^2}{\tau^2}\right) \tag{9.33}$$

and the fourier modes of the product of two elements of the basis can also be computed (see appendix C, equation C.38), and are given by

$$\langle n,0 \mid \mathcal{U}_1(k_1)\mathcal{U}_2(k_2) \mid n-1,0\rangle = (-)^{k_1 k_2} e^{-\xi/2}\sqrt{\frac{\pi\tau}{q}}\left(k_1 - \imath\frac{k_2}{\tau}\right)\sum_{j=0}^{n-1}(-)^j\frac{\sqrt{n!(n-1)!}\xi^j}{j!(n-1-j)!(j+1)!} \tag{9.34}$$

this allows the computation of the value of the integral

$$I^{(1)}_{n,n-1} = \frac{\pi\imath\tau}{q}\sqrt{1-\epsilon}\sum_{\vec{k}}\left[k_1^2 - \frac{k_2^2}{\tau^2} + 2\imath\frac{k_1 k_2}{\tau}\right]\frac{e^{-\xi}}{\xi}\sum_{j=0}^{n-1}(-)^j\frac{\sqrt{n!(n-1)!}\xi^j}{j!(n-1-j)!(j+1)!} \tag{9.35}$$

and the analytic determination of the contribution to the eigenvalue

$$\lambda^{(1)}_{np} = p\frac{\pi\tau}{q}\sqrt{1-\epsilon}\sum_{\vec{k}}\left[k_1^2 - \frac{k_2^2}{\tau^2}\right]\frac{e^{-\xi}}{\xi}\sum_{j=0}^{n-1}(-)^j\frac{\sqrt{n!(n-1)!}\xi^j}{j!(n-1-j)!(j+1)!} \tag{9.36}$$

note that the value of this contribution is zero for $\tau = 1$.

Higher order corrections

In general it is possible but difficult to obtain closed expressions for higher order corrections in the perturbative series. The analytical expression for the next corrections is

$$\lambda^{(2)}_{np} = \langle\psi^{(0)}_{np} \mid \not{D}^{(2)} \mid \psi^{(0)}_{np}\rangle + \sum_{n'\neq n;\pm'}\frac{\left|\langle\psi^{(0)}_{np} \mid \not{D}^{(1)} \mid \psi^{(0)}_{n'p'}\rangle\right|^2}{\lambda^{(0)}_{np} - \lambda^{(0)}_{n'p'}} \tag{9.37a}$$

$$\begin{aligned}\lambda^{(3)}_{np} &= \langle\psi^{(0)}_{np} \mid \not{D}^{(3)} \mid \psi^{(0)}_{np}\rangle + \\ &+ \sum_{n'p'}\frac{\langle\psi^{(0)}_{np} \mid \not{D}^{(1)} \mid \psi^{(0)}_{n'p'}\rangle\langle\psi^{(0)}_{n'p'} \mid \not{D}^{(2)} \mid \psi^{(0)}_{np}\rangle}{\lambda^{(0)}_{np} - \lambda^{(0)}_{n'p'}} + \\ &+ \sum_{n'p'}\frac{\langle\psi^{(0)}_{np} \mid \not{D}^{(2)} \mid \psi^{(0)}_{n'p'}\rangle\langle\psi^{(0)}_{n'p'} \mid \not{D}^{(1)} \mid \psi^{(0)}_{np}\rangle}{\lambda^{(0)}_{np} - \lambda^{(0)}_{n'p'}} + \\ &+ 2\langle\psi^{(0)}_{np} \mid \not{D}^{(1)} \mid \psi^{(0)}_{np}\rangle\sum_{n'p'}\frac{\left|\langle\psi^{(0)}_{np} \mid \not{D}^{(1)} \mid \psi^{(0)}_{n'p'}\rangle\right|^2}{\left(\lambda^{(0)}_{np} - \lambda^{(0)}_{n'p'}\right)^2}\end{aligned} \tag{9.37b}$$

all these contributions can be written in terms of the integrals $I^{(k)}_{n,n'}$, using the results of the previous section

$$\lambda^{(2)}_{np} = p\mathrm{Im}\left(I^{(2)}_{n,n-1}\right) + \frac{1}{4}\sum_{n',p'} \frac{\left|I^{(1)}_{n,n'-1} - \frac{p}{p'}\overline{I}^{(1)}_{n-1,n'}\right|^2}{\lambda^{(0)}_{np} - \lambda^{(0)}_{n'p'}} \tag{9.38a}$$

$$\begin{aligned}\lambda^{(3)}_{np} &= p\mathrm{Im}\left(I^{(3)}_{n,n-1}\right) + \\ &+ \frac{pp'}{4}\sum_{n'p'} \frac{I^{(1)}_{n,n'-1}\overline{I}^{(2)}_{n',n-1} + I^{(1)}_{n-1,n'}\overline{I}^{(2)}_{n'-1,n} - \frac{p}{p'}\left(I^{(1)}_{n-1,n'}\overline{I}^{(2)}_{n',n-1} + I^{(2)}_{n'-1,n}\overline{I}^{(1)}_{n,n'-1}\right)}{\lambda^{(0)}_{np} - \lambda^{(0)}_{n'p'}} + \\ &+ \frac{pp'}{4}\sum_{n'p'} \frac{I^{(2)}_{n,n'-1}\overline{I}^{(1)}_{n',n-1} + I^{(2)}_{n-1,n'}\overline{I}^{(1)}_{n'-1,n} - \frac{p}{p'}\left(I^{(2)}_{n-1,n'}\overline{I}^{(1)}_{n',n-1} + I^{(1)}_{n'-1,n}\overline{I}^{(2)}_{n,n'-1}\right)}{\lambda^{(0)}_{np} - \lambda^{(0)}_{n'p'}} + \\ &+ \frac{p}{2}\mathrm{Im}\left(I^{(1)}_{n,n-1}\right)\sum_{n'p'} \frac{\left|I^{(1)}_{n,n'-1} - \frac{p}{p'}\overline{I}^{(1)}_{n-1,n'}\right|^2}{\left(\lambda^{(0)}_{np} - \lambda^{(0)}_{n'p'}\right)^2}\end{aligned} \tag{9.38b}$$

Ten

Conclusions†

10.1 Summary of results

10.1.1 Solutions of the abelian Higgs models.

Let us summarise our results. We have analysed the Bogomolny equations of the abelian Higgs model in the two dimensional torus, and characterised all the solutions. We have proved that the moduli space of solutions of these equations has a fibre bundle structure, with fibre $\mathbb{CP}^{q-1}$ where q is the (topological) flux number. We have proved that there exist a one to one correspondence between the coordinates of the moduli space, and the positions of the vortex strings: just as in the case of vortices living in $\mathbb{R}^2$ the solutions with flux q can be interpreted as q vortices located at arbitrary (not necessarily distinct) points in the plane. Furthermore, and this is one of our central results, we have shown that one can expand the solutions of the Bogomolny equations in powers of the *Bradlow parameter*, $\epsilon = 1 - 2f$, where f is the average magnetic field (flux over area). This parameter interpolates between the minimum area for which the Bogomolny equations have solutions ($\epsilon = 0$), and the case of vortices living in the plane ($\epsilon = 1$).

The first order of this construction is explicitly given, and this approximate solution, valid when the torus area is close to the "critical" size ($\mathcal{A} = 4q\pi$), can be used to investigate some qualitative properties of the solutions of the Bogomolny equations. The coefficients of the Fourier modes for higher orders can be constructed using an iterative procedure involving convolutions. Although closed analytical expression for the coefficients can be constructed beyond the first non-trivial order, it is hard to reach a high order in this way. Usually is much more practical to construct them numerically. These coefficients can be determined up to double precision machine accuracy (15-17 significant digits), by truncating to a finite number of modes. This method can be applied to construct any solution, with arbitrary flux and location of the Higgs field zeroes (i.e. we can obtain any solution of the moduli space). This property of our method is essential, since most of the other methods are specific to determine solutions with a certain symmetry (i.e. solutions with cylindrical symmetry). Our method

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is able to provide any solution, with no difference in performance when studying multivortex configurations.

With the help of this code we have computed the coefficients for a couple of cases ($q = 1$ and $q = 2$) and the results are very encouraging. The 51 order truncated expansion is estimated to describe the shape of the function within machine precision up to areas which are two and a half times the critical area. But meaningful values can be extracted also for large sizes, where due to the exponential localisation of the solution, the configurations are close to those of infinite area. In particular, these results match nicely with what is known about the unit-vortex on the plane. Turning the information around, this allows to obtain precise expectations about the behaviour of the coefficients for large order, which are satisfied by the data.

A simple generalisation of the abelian Higgs model with a global $SU(N)$ symmetry, known as the extended abelian Higgs model, has deep implications in the phenomenology of cosmic strings. These models arise naturally as the bosonic part of supersymmetric theories or strings theories, and have vortex-like solutions, given as the solutions of the Bogomolny equations. This extended abelian Higgs model in the two dimensional torus has also been investigated. The moduli space of solutions of the Bogomolny equations has been characterised: it also has a fibre bundle structure, with fibre given by $\mathbb{CP}^{Nq-1}$. After having characterised the moduli space, and shown that the Bradlow parameter expansion can be used to obtain the solutions we have applied the method to construct explicit solutions for the special case $N = 2$, since this is the case that has been studied in more detail for the case of vortices living in $\mathbb{R}^2$. This time the results for vortices living in the torus are qualitative very different from those of vortices living in the plane. The moduli space of solutions for extended vortices with flux number q living in $\mathbb{R}^2$ can be interpreted as the locations of the vortex strings with a certain width. The width of the vortex string is an extra parameter that appears in the extended models. On the other hand vortices in the torus have a very different look. They are certainly vortices with variable size, but the relation between the coordinates of the moduli and the positions/widths of this objects is unclear. Even worse, some solutions seem to have more vortices than the flux number. We have constructed solutions in the torus with flux number $q = 2$ with four structures, each carrying a flux $1/2$. This amazing results resemble the study of self dual configurations in the four dimensional torus, where objects with fractional topological charge exist. This makes natural the interpretation of the moduli as made of objects with fractional topological charge. These results can be of vital importance for people doing numerical computations, since periodic boundary conditions are a natural choice.

We have been able to solve the Bogomolny equations also in this case. The technical procedure to solve the Bogomolny equations is very similar as in the previous case. We have shown that the solutions of the extended abelian Higgs model in the torus with flux number $q = 1$ are non extended vortices. For the case of vortices living in $\mathbb{R}^2$ there exist a one parameter family of solutions, and this parameter can be interpreted as the size of the object and its orientation in the internal space, but this one parameter family of solutions does not exist for vortices living in the torus. We have explored in detail the moduli space of solutions for the case $N = 2$ and $q = 2$. In this case a general solution to the Bogomolny equations exists given any three real parameters. We have shown how a dissociation process exist for the case of extended vortices living in the torus: in spite of the fact that the flux number is $q = 2$, and that for the case of vortices living in $\mathbb{R}^2$ the solutions are well described by two vortices, we can see, for some values of the coordinates of the moduli, four structures. The points where this dissociation process takes place has been quantified for some kind of solutions of the Bogomolny equations.

To reproduce the $q = 1$ solution of an extended vortex with variable size living in $\mathbb{R}^2$, we have used a $q = 2$ configuration in the torus. Computing 51 orders of the expansion, and taking the infinite area limit in an appropriate way we have shown that we can reproduce the configurations in $\mathbb{R}^2$.

The solutions on the torus are relevant to depict the behaviour of the system in a situation of high vortex density. Their thermodynamics was analysed in Ref. [72]. The description in terms of Fourier modes has been used previously in other contexts, like in the study of skyrmion crystals Ref. [81, 82]. It seems, however, on the basis of our results, that our method can be used successfully to study the infinite area case as well.

`FORTRAN 90` (free) code that implements these algorithms, for obtaining both the solutions of the extended and non extended abelian Higgs model is available at

`http://lattice.ft.uam.es/perpag/alberto/codigo_en.php`

10.1.2 Dynamics of vortices

In the case of low energy motion of vortices, their dynamics can be described by a geodesic motion in the moduli space of solutions of the Bogomolny equations. The difficult part to describe this dynamics is to obtain the metric in the moduli space. Again the non existence of analytical solutions to the Bogomolny equations makes harder to compute the metric, and here is where the semi-analytical character of our method shows its power: We can expect to obtain an analytic approximation to the metric valid for areas close to the critical one, and a formal expansion for the metric in powers of the Bradlow parameter based in the expansion for the fields. We have shown that it is possible to obtain the metric as a power series in the Bradlow parameter ϵ. To any finite order in the expansion the metric has a known form depending on a small number of real parameters. We give their analytic expression up to second order for arbitrary number of vortices q. The leading term, giving the Bradlow limit result, is proportional to the Fubini-Study metric. This coincides with results obtained for the two-sphere [83]. The second order result depends on q^2 constants, which are given in the form of rapidly decreasing double infinite sums. Although increasingly complicated it is not too hard to go beyond this order. Rather than attempting this for the general vortex case, we have concentrated on the two-vortex case. We have given the form of the metric up to 5th order and the numerical value of their 15 real coefficients with 14-16 significant digits (machine double precision). Furthermore, it is possible to go much beyond this order in the computation of the metric at specific points of the moduli space by means of a computer. Essentially, the implementation is based on the truncation of the infinite sums to finite ones. In this way we have gone up to 40th order with fairly limited computational effort. Furthermore, the fast convergence of the sums stills maintains the double machine precision accuracy in the determination of the metric. Having so many terms in the expansion has allowed us to extrapolate our results to the infinite area case (vortices on the plane) with quite satisfactory results. The resulting metric is consistent with all the previously known results, both analytical in certain limits and numerical. The result have fairly small and controlled errors arising from the extrapolation and the truncation in the expansion.

The computation of the metric allows the study of vortex scattering, known to be a good approximation up to relatively high vortex velocities. The presence of the torus can be used to mimic the behaviour of vortices in a dense environment. A natural question that poses itself is the possibility that two-vortex dynamics on the torus is integrable. This is actually the case for both small (close to critical) torus sizes as for infinitely large ones (the plane). We have studied Poincare maps as a first step in this study. Furthermore, we emphasise that, as in our previous work, our method has been found to be a competitive tool to study the properties of vortices on the plane. It has important advantages over other methods: its semi-analytical nature and the possibility of dealing with an arbitrary number of vortices. In this work, for example, we have given the curves (with errors) that describe the functional dependence of the two-vortex scattering angle with the impact parameter.

The moduli metric has also been studied for the case of the non extended abelian Higgs model. We have obtained an expression for the metric, and again this expression allows us to write the metric as a power expansion in the Bradlow parameter. The first order of this expansion has been analytically obtained. Higher orders in the expansion can be obtained with the help of a computer code. This is part of a work in progress.

The `FORTRAN 90` (free) code that implement the algorithm to compute the metric, and that has been used to obtain the numerical results of this work, is available at

`http://lattice.ft.uam.es/perpag/alberto/codigo_en.php`

10.1.3 Fermions in the presence of vortex solutions

As part of a work in progress, we have presented another application of the Bradlow parameter expansion. We have studied the behaviour of fermionic fields under the presence of a vortex solution in the torus. We have proved that the eigenvalues and eigenfunctions of the Dirac operator can be obtained in a systematic way, as a power expansion in the Bradlow parameter.

We have given an analytic expression for the first order in the expansion, and we have computed this correction in some representative examples. Higher order corrections must be computed using numerical methods, that are being programmed at this moment.

10.2 Further applications of the work

As we have said, the semi-analytical character of our method is one of its more powerful characteristics. The reason is that to study many properties of soliton solutions, it is in general necessary to know the classical solution. Although a purely numerical method, like a numerical integration of the partial differential equations, is also able to give the solutions, you will get only that: a numerical estimation of the solution. On the other hand our semi-analytical method, and the possibility of obtaining closed analytical expressions for the first order of the expansion, that can be considered as an approximate solution for sizes near of the critical, opens doors to investigate more complex problems. It is very likely that you can solve other interesting problems not covered in this work with the help of the Bradlow parameter expansion. A concrete example under study is the computation of the quantum corrections to the vortex solution, where the same methodology that we used in this work to compute the eigenvalues and eigenfunctions of the Dirac operator should be used.

The method of the Bradlow parameter expansion is generalisable to other manifolds where the Bradlow parameter can be introduced [57]. It can be any two dimensional compact manifold, or even abelian or non abelian gauge-Higgs theories in an arbitrary number of dimensions. As a matter of fact, the first order of the metric in the Bradlow parameter expansion has been computed for the case of vortices living in the two sphere [83].

There are a number of possible applications and generalisations of the method to other problems or situations, some of which are currently under study. Special mention deserves the application to self-dual configurations on the four dimensional torus. As we have seen in the introduction, there are no explicit analytic formulas for the solutions and the present method might provide good results, as an alternative to purely numerical methods [84, 31].

The method can be generalised beyond. For example it has been applied recently (see [85, 86]) to the study of abelian vortices in the non commutative torus.

Once

Conclusiones†

11.1 Resumen de resultados

11.1.1 Soluciones del model Higgs abeliano.

Hemos analizado las ecuaciones de Bogomolny del modelo Higgs abeliano en el toro bidimensional, y caracterizado todas sus soluciones. Hemos demostrado que la variedad de soluciones de estas ecuaciones tiene una estructura de fibrado, donde la fibra es el espacio complejo proyectivo $\mathbb{CP}^{q-1}$ (q es el flujo o carga topológica). Hemos demostrado que existe una correspondencia uno a uno entre las coordenadas que parametrízan la variedad de soluciones de las ecuaciones de Bogomolny y la posición de los vórtices: al igual que en el caso de los vórtices en el plano, las soluciones con flujo q se pueden interpretar como q vortices situados en cualesquiera posiciones (no necesariamente diferentes) posiciones en el plano. Mas aun, y este es uno de nuestros resultados centrales, hemos demostrado que todas estas soluciones se pueden obtener como una serie de potencias en el *parámetro de Bradlow*, $\epsilon = 1 - 2f$, donde f es el campo magnético medio que cruza nuestro toro (el flujo total dividido entre el área). Este parámetro interpola entre el tamaño mínimo que ha de tener nuestro toro para que las ecuaciones de Bogomolny tengan solución ($\epsilon = 0$), y el caso de los vórtices viviendo en el plano ($\epsilon = 1$).

El primer orden de esta expansión se ha hallado explícitamente, obteniendo así una solución analítica aproximada para áreas cercanas al área critica ($\mathcal{A} = 4q\pi$), que puede ser usada para investigar algunas propiedades cualitativas de las soluciones. Los coeficientes de Fourier para ordenes mas altos en la expansión pueden hallarse mediante un procedimiento iterativo. Aunque en principio se pueden construir expresiones analíticas para estos coeficientes para ordenes mas altos que el primero, es laborioso llegar hasta un orden alto de esta forma. Generalmente es mucho mas practico calcularlos con la ayuda de un ordenador. Estos coeficientes se pueden determinar hasta precision maquina (15-17 cifras significativas) truncando las series de Fourier a un numero finito de modos. Este método se puede aplicar para construir cualquier solución, independientemente de los valores del flujo o de la posición de los vórtices (es decir, que podemos obtener cualquier solucion de todo el posible espacio de soluciones).

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Esta característica de nuestro método es esencial, ya que la mayoría de los metodos alternativos para obtener las soluciones están específicamente diseñados para construir cierto tipo de soluciones, en general con un alto grado de simetría (por ejemplo, soluciones de simetría cilindrica). Nuestro método es capaz de hallar cualquier solución, sin un coste adicional en tiempo de computación para hallar soluciones multi-vortice.

Con la ayuda de este código hemos calculado los coeficientes de Fourier para un par de casos representativos ($q = 1$ y $q = 2$) y los resultados son muy prometedores. La expansión truncada a 51 ordenes es capaz de describir las soluciones en el toro hasta precisión maquina para áreas dos veces y media mayores que el área crítica. Se pueden obtener valores para áreas mayores que la mencionada, debido a la naturaleza exponencialmente localizada de la solución las configuraciones son muy parecidas a las configuraciones en el plano. En particular nuestros resultados concuerdan muy bien con lo que se conoce sobre el vórtice de flujo uno en el plano. Utilizando esa información hemos sido capaces de predecir el comportamiento de los coeficientes para ordenes grandes, concordando con los datos numéricos.

Una sencilla generalización del modelo Higgs abeliano con una simetría global $SU(N)$, conocida como el modelo Higgs abeliano extendido, tiene profundas implicaciones en la fenomenología de las cuerdas cósmicas. Estos modelos surgen de forma natural como la parte bosonica de teorías supersimetricas, y también tienen soluciones tipo vórtice, que se obtienen solucionando las correspondientes ecuaciones de Bogomolny. También hemos investigado Este modelo Higgs abeliano extendido en el toro bidimensional. De nuevo hemos caracterizado el espacio de soluciones: también tiene una estructura tipo fibrado, donde la fibra es de nuevo el espacio proyectivo complejo $\mathbb{CP}^{Nq-1}$. Tras haber caracterizado el espacio de soluciones y demostrado que la expansión en el parámetro de Bradlow se puede usar también aquí para obtener todas las soluciones, nos hemos concentrado en el estudio del modelo con $N = 2$, ya que este es el caso que mas en detalle se ha estudiado en la literatura para el caso de vórtices viviendo en $\mathbb{R}^2$. En esta ocasión los resultados que obtenemos son cualitativamente distintos para el caso de vórtices viviendo en $\mathbb{R}^2$ y en el toro. Las soluciones tipo vórtice viviendo en el plano de flujo q se pueden interpretar como q vórtices de tamaño y orientación en el espacio interno arbitrarios. El tamaño del vórtice es un parámetro nuevo genuino de los modelos extendidos. por otro lado las soluciones en el toro son bien distintas. Desde luego también tienen el aspecto de vórtices, pero en este caso la relación entre las coordenadas del espacio de soluciones y la posición/tamaño/orientación de los vórtices dista de estar clara. Aun peor, para ciertos valores de las coordenadas del espacio de soluciones, hay mas vórtices que carga topológica. Hemos construido explícitamente soluciones en el toro de flujo $q = 2$ con 4 estructuras, cada una portadora de un flujo 1/2. Este sorprendente resultado recuerda al estudio de las configuraciones auto-duales en el toro tetra-dimensional, donde existen objetos de carga fraccionaria. Estos resultados hacen que sea natural interpretar las soluciones de las ecuaciones de Bogomolny como construcciones hechas con objetos de carga fraccionaria. Estos resultados pueden ser muy importantes para la gente que esta realizando simulaciones numéricas, ya que las condiciones de contorno periódicas son comunes en este ámbito.

En este caso también hemos sido capaces de resolver las ecuaciones de Bogomolny. El procedimiento es muy similar al caso anterior. Hemos comprobado que todas las soluciones al modelo Higgs abeliano extendido con flujo $q = 1$ son vortices comunes (no extendidos). En el caso del modelo Higgs abeliano extendido en el plano, existe una familia uniparamétrica de soluciones, donde el parámetro puede asociarse al tamaño del vórtice y a su orientación en el espacio interno, pero esta familia uniparamétrica no existe para el caso de vórtices viviendo en el toro. Hemos explorado en detalle el espacio de soluciones con $N = 2$ y $q = 2$. En este caso una solución arbitraria de las ecuaciones de Bogomolny viene dada por cinco parámetros reales. Hemos comprobado que existe un proceso de disociación para el caso de vórtices extendidos en el toro: a pesar de que el flujo es $q = 2$, y de que en el caso de vórtices viviendo en el plano las soluciones se entienden correctamente en términos de 2 estructuras, podemos ver, para algunos valores de las coordenadas del espacio de soluciones, cuatro estructuras. Los puntos en los que la disociación tiene lugar, se han caracterizado para algunas clases de soluciones.

Para reproducir la solución en el plano con flujo $q = 1$, hemos usado una configuración en el toro con $q = 2$. Calculando 51 ordenes en la expansión, y tomando el limite de area infinita de un modo apropiado, hemos reproducido las configuraciones en el plano.

Las soluciones en el toro para representar un sistema con una alta densidad de vórtices. Su termodinámica ha sido analizada en la referencia [72]. La descripción de este tipo de soluciones en termino de sus modos de Fourier ha sido utilizada anteriormente en otros contextos, como el estudio de los “skyrmion cristals” Ref. [81, 82]. En base a nuestros resultados podemos decir que el método no es solo útil para estudiar estos casos, sino también para estudiar vórtices en el plano.

La implementación de estos algoritmos para obtener las soluciones tanto de los modelos extendidos como no extendidos en `FORTRAN 90` esta disponible como código libre en

`http://lattice.ft.uam.es/perpag/alberto/codigo_en.php`

11.1.2 Dinamica de vortices

En el caso de vórtices moviéndose a velocidades no relativistas, su dinámica se puede describir como un movimiento geodésico en el espacio de soluciones de las ecuaciones de Bogomolny. La parte difícil de describir esta dinámica consiste en obtener la métrica que determina las geodésicas. De nuevo, la falta de soluciones conocidas a las ecuaciones de Bogomolny hace mas difícil calcular la me’trica, y aquí es donde el caracter semi-analítico de nuestro método muestra su poder: podemos obtener una aproximación a la métrica para áreas cercanas al área critica, y una expansión formal para la métrica en potencias del parámetro de Bradlow basada en la expansión de los campos. Hemos demostrado que es posible obtener la métrica como una serie de potencias en el parámetro de Bradlow ϵ. Para cualquier orden finito en la expansión la métrica depende de un puñado de números. Los números correspondientes a los dos primeros ordenes se han calculado para cualquier valor arbitrario de la carga topológica q. El primer termino es proporcional a la métrica de Fubini-Study, coincidiendo con los resultados previamente obtenidos para vórtices viviendo en la esfera [83]. El segundo orden depende de q^2 constantes, que se dan en forma de dobles sumas infinitas rápidamente decrecientes. A pesar de que es mas complicado obtener expresiones para ordenes mas altos, es factible calcular algún orden mas. Nosotros nos hemos concentrado en el problemas mas interesante: el caso de la dinámica de dos vórtices. Hemos calculado la forma de la métrica hasta quinto orden, y los valores numéricos (a precisión maquina) de los 15 parámetros reales de los que depende. De hecho es posible calcular muchos mas ordenes en algunos puntos específicos mediante un ordenador. La implementación esta básicamente basada en truncar las sumas infinitas a un numero finito de términos. De esta forma hemos llegado a calcular 40 ordenes con un coste de computación reducido. Mas aun, la rápida convergencia de las dobles sumas, hace que la métrica se pueda determinar con precisión maquina. Teniendo tantos ordenes en la expansión del parámetro de Bradlow, hemos podido extrapolar los resultados al caso de área infinita (vórtices en el plano) con resultados muy satisfactorios. La métrica resultante es consistente con todos los resultados previamente conocidos sobre ella, tanto analíticos en ciertos limites, como numéricos. El resultado tiene errores pequeños y controlados que vienen de la extrapolación y de la truncación de la expansión.

El calculo de la métrica nos permite estudiar la colisión de vórtices con una buena aproximación si las velocidades de colisión no son muy grandes. La presencia del toro puede usarse para imitar el comportamiento de los vórtices en un medio muy denso. Una cuestión interesante es si la dinámica de dos vórtices en el toro es integrable. Este es el caso tanto para áreas muy pequeñas (cercanas a la critica), como para el caso de área infinita. Como primer paso para examinar esta cuestión hemos estudiado los mapas de Poincare. Una vez mas queremos resaltar el hecho de que nuestro método es una herramienta competitiva para estudiar propiedades de los vórtices en el plano. Tiene importantes ventajas sobre otros métodos: su carácter semi-analitico hace posible el estudio de un numero arbitrario de vórtices. En este trabajo, por ejemplo se han calculado las con errores de la dependencia del ángulo de dispersión con el parámetro de impacto.

La métrica en el espacio de soluciones también se ha estudiado para el caso de vórtices del modelo Higgs abeliano extendido. Hemos obtenido una expresión para la métrica, y de nuevo nos ha permitido escribir la métrica como una serie de potencias en el parámetro de Bradlow. El primer orden ha sido calculado analíticamente. Ordenes mas altos se pueden calcular con ayuda de un ordenador, trabajo que esa realizándose en el momento de escribir esta tesis.

La implementación de estos algoritmos para obtener las soluciones tanto de los modelos extendidos como no extendidos en `FORTRAN 90` esta disponible como código libre en

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11.1.3 Fermions in the presence of vortex solutions

Como parte de un trabajo en curso, hemos presentado otra aplicación de la expansión en el parámetro de Bradlow. Hemos estudiado el comportamiento de los fermiones en presencia de soluciones tipo vórtice. Hemos demostrado que las autofunciones y los autovalores del operador de Dirac se pueden obtener de una manera sistemática como una expansión en serie de potencias en el parámetro de Bradlow.

Hemos dado una expresión analítica para el primer order en la expansión, y hemos calculado esta corrección en algún caso representativo. Ordenes mas altos pueden ser calculados con ayuda de un ordenador, algo que esta siendo programado en estos momentos.

11.2 Aplicaciones del método

Como hemos dicho, la naturaleza semi-analitica de nuestro métodos es una de sus características mas poderosas. La razón es que para estudiar muchas propiedades de los solitones, uno necesita saber la solución clásica. Aunque un método puramente numérico, como la integración numérica de las ecuaciones diferenciales en derivadas parciales, es capaz de obtener las soluciones, solo tendremos eso: Una estimación numérica de la solución. Por otro lado nuestro metodo semi-analitico, y la posibilidad de obtener soluciones aproximadas de una forma analítica exacta, abre las puertas a investigar problemas mas complejos. Es muy probable que numerosos problemas interesantes que no se han tratado en este trabajo se puedan afrontar con el metodo de la expansión en el parámetro de Bradlow. Un ejemplo concreto que esta siendo estudiado es el calculo de las correcciones cuánticas a las soluciones tipo vórtice, donde hay que usar la misma metodología empleada para hallar las autofunciones y autovalores del operador de Dirac.

El método de la expansión en el parámetro de Bradlow es generalizable a otras variedades donde un parámetro de Bradlow tenga sentido [57]. Puede ser cualquier superficie bidimensional compacta, o incluso teorías gauge abelianas o no abelianas acopladas a un Higgs en un numero arbitrario de dimensiones. De hecho el primer orden en el esta expansión para la métrica se calculó por primera vez para vortices viviendo en la esfera bidimensional [83].

Hay numerosas aplicaciones y generalizaciones a otros problemas o situaciones, algunas de las cuales están siendo estudiadas en este momento. Una mención especial merece la aplicación al calculo de configuraciones autoduales en el toro tetradimensional. Como hemos comentado en la introducción, no hay soluciones explicitas conocidas en este caso, y nuestro método puede dar buenos resultados, como alternativa a los métodos puramente numéricos [84, 31].

El método se puede extender mas allá. Por ejemplo se ha aplicado recientemente (ver [85, 86]) al calculo de vórtices abelianos en el toro no conmutativo.

A

Conventions

It is important to note that these are conventions, not rules carved in stone.

Wikipedia naming conventions

A.1 Minkowski and Euclidean spaces

In some parts of this thesis, we work in $3+1D$ space time, and in others in $2+1$. Greek letters of the end of the alphabet $(\mu, \nu, \rho, \dots)$, will run over the minkowskian space time indices. The metric of this space will be

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{A.1}$$

and an analogous metric in the case of $3+1D$. The exact total number of dimensions (three or four), should be clear from the context. Greek letters from the beginning of the alphabet $(\alpha, \beta, \dots)$ will run over the Euclidean indices In this case the metric is simply given by $g_{\alpha_\beta} = \delta_{\alpha_\beta}$.

Latin letters $(i, j, k, \dots)$ will run only over spatial coordinates, and latin letters from the begining of the alphabet like $a, b, c, \dots = 1, 2, 3$ will run over internal symmetries (like the $SU(2)$ colour symmetry of QCD).

A.2 Pauli matrices and $SU(2)$ generators

The pauli matrices are defined as τ_a, and are given by:

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.2}$$

They are hermitian traceless matrices, and they obey the relations:

$$[\tau_a, \tau_b] = 2i\epsilon_{abc}\tau_c \quad \{\tau_a, \tau_b\} = 2\delta_{ab}\mathbb{I} \quad \mathrm{tr}(\tau_a\tau_b) = 2\delta_{ab} \tag{A.3}$$

The generators of the $SU(2)$ algebra are defined as $\frac{\tau_a}{2}$. We will use in some parts of the text the "ladder" operators $\tau_\pm$:

$$\tau_\pm = \frac{1}{2}\left(\tau_1 \pm \imath\tau_2\right) \tag{A.4}$$

That are given by

$$\tau_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \tau_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{A.5}$$

This "ladder" operators obeys the commutation relations

$$\left[\tau_\pm, \frac{\tau_3}{2}\right] = \pm\tau_\mp \quad \left[\tau_+, \frac{\tau_-}{2}\right] = \frac{\tau_3}{2} \quad \left\{\tau_\pm, \frac{\tau_3}{2}\right\} = \mathbb{O} \tag{A.6}$$

and the properties

$$(\tau_\pm)^2 = \mathbb{O} \quad \mathrm{tr}(\tau_+\tau_-) = \mathrm{tr}(\tau_-\tau_+) = 1 \quad \mathrm{tr}(\tau_\pm\tau_3) = 0 \tag{A.7}$$

An element of the $SU(2)$ algebra is given by ($g_a \in \mathbb{R}$)

$$\mathbf{g} = g_a\frac{\tau_a}{2} = g\frac{\tau_3}{2} + g^+\tau_+ + g^-\tau_- \tag{A.8}$$

Where $g = g_3$ and $g^+ = (g^-)^* = \frac{1}{2}(g_1 - \imath g_2)$ (the $*$ denote the complex conjugate).

The trace of the product of two $SU(2)$ elements is given by:

$$\mathrm{tr}(\mathbf{gh}) = g_a h_b \mathrm{tr}(\tau_a\tau_b) = \frac{1}{2}g_i h_i = \frac{1}{2}gh + g^+h^- + g^-h^+ \tag{A.9}$$

When working in four dimensions with the $SU(2)$ group, there is an special and very useful notation, that we will define here. Using the Pauli matrices, defined above, we will define the symbols

$$\sigma_\alpha = (-\imath\tau_i, \mathbb{I}) \tag{A.10}$$
$$\overline{\sigma}_\alpha = (\imath\tau_i, \mathbb{I}) \tag{A.11}$$

These matrices obey the relation

$$\overline{\sigma}_\alpha\sigma_\beta = \overline{\eta}^\delta_{\alpha\beta}\sigma_\delta \tag{A.12}$$

Where $\overline{\eta}^\delta_{\alpha\beta}$ is one of the two 't Hooft eta symbols, defined by

$$\begin{aligned} \overline{\eta}^4_{\alpha\beta} &= \delta_{\alpha\beta} \\ \overline{\eta}^k_{ij} &= \varepsilon^{kij}; \quad \overline{\eta}^k_{i4} = -\overline{\eta}^k_{4i} = -\delta^k_i; \quad \overline{\eta}^k_{44} = 0 \end{aligned} \tag{A.13}$$
$$\begin{aligned} \eta^4_{\alpha\beta} &= \delta_{\alpha\beta} \\ \eta^k_{ij} &= -\varepsilon^{kij}; \quad \eta^k_{i4} = -\eta^k_{4i} = -\delta^k_i; \quad \eta^k_{44} = 0 \end{aligned} \tag{A.14}$$

The contraction of any vector with the symbol σ_α, will be simply denoted with the letter of the vector without any index i.e.

$$\begin{aligned} A_\alpha\sigma_\alpha &= A \\ A_\alpha\overline{\sigma}_\alpha &= \overline{A} \end{aligned} \tag{A.15}$$
$$\tag{A.16}$$

This can be done with any vector, including the derivative operators

$$\partial_\alpha \sigma_\alpha = \partial \quad (A.17)$$
$$\partial_\alpha \overline{\sigma}_\alpha = \overline{\partial} \quad (A.18)$$
$$D_\alpha \sigma_\alpha = D \quad (A.19)$$
$$D_\alpha \sigma_\alpha = \overline{D} \quad (A.20)$$

In the same way, two dimensional

A.3 Dirac Matrices

In $2+1D$ we will define the Dirac matrices γ^μ as

$$\gamma^0 = \tau_3; \quad \gamma^1 = \imath\tau_2; \quad \gamma^2 = -\imath\tau_1 \quad (A.21)$$

A.4 Complex notation

In general, when we work with two dimensional fields, it is very useful to write them as functions in the complex plane, so that we can use all the power of the complex analysis. If a point in the plane (or the torus or any two dimensional manifold) is represented by (x_1, x_2), we can label it also by a complex coordinate z and its complex conjugate $\overline{z}$, defined by

$$z = \frac{x_1 + \imath x_2}{2}; \qquad \overline{z} = \frac{x_1 - \imath x_2}{2} \quad (A.22a)$$

the inverse relations are

$$x_1 = z + \overline{z}; \qquad x_2 = -\imath(z - \overline{z}) \quad (A.22b)$$

we will also define partial derivatives

$$\partial = \partial_1 - \imath\partial_2; \qquad \overline{\partial} = \partial_1 + \imath\partial_2 \quad (A.22c)$$

and the inverse relations

$$\partial_1 = \frac{1}{2}(\partial + \overline{\partial}); \qquad \partial_2 = \frac{1}{2\imath}(\partial - \overline{\partial}) \quad (A.22d)$$

The notation is chosen so that $\partial z = \overline{\partial}\overline{z} = 1$. In general we will indicate complex conjugate with a line over the quantity. A general function of two variable $f(x_1, x_2)$ is translated in a function of $(z, \overline{z})$, and real vector fields $v_i(x_1, x_2)$ are grouped in a single complex function of two variables

$$v(z, \overline{z}) = v_1(x_1(z, \overline{z}), x_2(z, \overline{z})) - \imath v_2(x_1(z, \overline{z}), x_2(z, \overline{z})) \quad (A.23)$$

B

Doubly periodic Functions and the Jacobi theta functions

> When I was a student, abelian functions were, as an effect of the Jacobian tradition, considered the uncontested summit of mathematics, and each of us was ambitious to make progress in this field. And now? The younger generation hardly knows abelian functions.
>
> Felix Klein

We will structure this appendix as a summary of results (theorems) with their proofs. They will show the most important properties of the theta functions for us.

First we begin with the study of doubly periodic functions, and we will see some important properties of them, later we will investigate the Jacobi theta functions, that are quasi-double periodic functions. There we will need the results obtained in the study of the doubly periodic functions.

This appendix is self contained, and no prior knowledge of elliptic functions or theta functions is assumed.

B.1 Double periodic functions.

Definition B.1.1 *A complex analytic function $f(z)$ is called double periodic iif there exist two complex numbers w_1, w_2 $(w_1/w_2 \notin \mathbb{R})$, such that:*

$$f(z + 2w_1) = f(z + 2w_2) = f(z) \tag{B.1}$$

A doubly periodic function that is analytic (except at poles), and that have no singularities except that poles is called an elliptic function.

We are interested in double periodic analytic functions in the complex plane. The first *really* important result for us is:

Theorem B.1.1 *The only elliptic functions without poles are constant.*

Proof B.1.1 *Since the torus is compact and the function is analytic, the function must be bounded. Then by Liouville's theorem all analytic bounded functions on the complex plane are constant*

We will call a *cell* the paralelogram obtained joining the points $t, t+2w_1$, $t+2w_1+2w_2$, $t+2w_2$, with t chose in such a way that there are no poles in the paralelogram. Clearly this is always possible thanks to the next theorem.

Theorem B.1.2 *The numbers of poles of an elliptic function in any paralelogram $t, t+2w_1$, $t+2w_1+2w_2$, $t+2w_2$ is finite*

Proof B.1.2 *If not the set of points must have a limit point (since the paralelogram is compact), and this limit point is an essential singularity, but elliptic functions do not have essential singularities.*

Obviously from the definition of an elliptic function, the values of an elliptic function in any cell are a mere repetition of their values in any other cell.

Theorem B.1.3 *The numbers of zeros of an elliptic function (except $f(z)=0$) in any cell is finite.*

Proof B.1.3 *If not the inverse function, that is also an elliptic function, would have an infinite number of poles in the cell.*

The next result will be useful in the future.

Theorem B.1.4 *The sum of the residues of an elliptic function, $f(z)$, at its poles in any cell is zero.*

Proof B.1.4 *Let $\Box$ represent the contour formed by the edges of the cell, and let the corners of the cell be $t, t+2w_1, t+2w_1+2w_2, t+2w_2$. Then the sum of residues of $f(z)$ at its poles in $\Box$ is:*

$$\frac{1}{2\pi i}\int_{\Box} f(z)\mathrm{d}z = \frac{1}{2\pi i}\left\{\int_t^{t+2w_1} + \int_{t+2w_1}^{t+2w_1+2w_2} + \int_{t+2w_1+2w_2}^{t+2w_2} + \int_{t+2w_2}^{t}\right\} f(z)\mathrm{d}z \qquad \text{(B.2)}$$

Now we can write in the second and third integrals $z+2w_1$ and $z+2w_2$ respectively for z, and then we have:

$$\frac{1}{2\pi i}\int_t^{t+2w_1}[f(z)-f(z+2w_2)]\mathrm{d}z - \frac{1}{2\pi i}\int_t^{t+2w_2}[f(z)-f(z+2w_1)]\mathrm{d}z \qquad \text{(B.3)}$$

That by the periodic properties of the elliptic functions is zero.

Definition B.1.2 *The number of poles of an elliptic function is called its* order.

Theorem B.1.5 *The order of the elliptic function $f(z)$ is equal to the numbers of roots of the equation:*

$$f(z) = c \qquad \text{(B.4)}$$

Proof B.1.5 *Since $f(z)$ is an elliptic function, $f(z)-c$ is also an elliptic function, and then*

$$\frac{1}{2\pi i}\int_{\Box}\frac{f'(z)}{f(z)-c}\mathrm{d}z \tag{B.5}$$

Is the numbers of zeros minus the numbers of roots of the equation $f(z)-c=0$, but since $f'(z+2w_1)=f'(z+2w_2)=f'(z)$, we can split the integral in four as before, and show that (B.5) is just zero.

We had shown that the only elliptic functions of order zero are constant. We cannot have elliptic functions of order one, and for elliptic functions of order two we have two possibilities: That the function has a double pole (for example the Weierstrass elliptic function $\wp(z)$) or that the function has two simple poles (for example the Jacobi elliptic functions $sn(z), cn(z), dn(z)$).

Much more information about this topic can be found in [6]

B.2 The Jacobi theta functions.

B.2.1 Definition of the four theta functions.

The Jacobi theta functions $\vartheta_i(z;q)=\vartheta_i(z|\tau)$ are dependent of the parameter $q=e^{\pi i\tau}$ and are defined by:

$$\vartheta_1(z;q) = \sum_{n=-\infty}^{\infty}(-)^{n-1/2}q^{(n+1/2)^2}e^{2(n+1/2)iz} \tag{B.6a}$$

$$\vartheta_2(z;q) = \sum_{n=-\infty}^{\infty}q^{(n+1/2)^2}e^{2(n+1)iz} \tag{B.6b}$$

$$\vartheta_3(z;q) = \sum_{n=-\infty}^{\infty}q^{n^2}e^{2niz} \tag{B.6c}$$

$$\vartheta_4(z;q) = \sum_{n=-\infty}^{\infty}(-)^n q^{n^2}e^{2niz} \tag{B.6d}$$

Which can also be written as:

$$\vartheta_1(z;q) = 2q^{1/4}\sum_{n=0}^{\infty}(-)^n q^{n(n+1)}\sin[(2n+1)z] \tag{B.7a}$$

$$\vartheta_2(z;q) = 2q^{1/4}\sum_{n=0}^{\infty}q^{n(n+1)}\cos[(2n+1)z] \tag{B.7b}$$

$$\vartheta_3(z;q) = 1+2\sum_{n=0}^{\infty}q^{n^2}\cos(2nz) \tag{B.7c}$$

$$\vartheta_4(z;q) = 1+2\sum_{n=0}^{\infty}(-)^n q^{n^2}\cos(2nz) \tag{B.7d}$$

This equations only have sense (i.e. the sums are convergent) for $|q|<1 \Longleftrightarrow Im(\tau)>0$. It is clear that in this situation the four Jacobi theta functions are analytic functions. It is easy to see that:

$$\vartheta_3(z|\tau) = \vartheta_4\left(z+\frac{\pi}{2}|\tau\right) \quad \text{(B.8a)}$$

$$\vartheta_1(z|\tau) = -ie^{iz+\frac{1}{4}\pi i\tau}\vartheta_4\left(z+\frac{\pi\tau}{2}|\tau\right) \quad \text{(B.8b)}$$

$$\vartheta_2(z|\tau) = \vartheta_1\left(z+\frac{\pi}{2}|\tau\right) \quad \text{(B.8c)}$$

B.2.2 Periodicity and zeros of the theta functions.

Now we are going to see one of the most important results for us. It is easy to check that ϑ_3 has the property:

$$\vartheta_3(z+\pi;q) = \vartheta_3(z;q) \quad \text{(B.9)}$$

It is also easy to check that:

$$\vartheta_3(z+\tau\pi;q) = \sum_{n=-\infty}^{\infty} q^{n^2}q^{2n}e^{2niz} = q^{-1}e^{-2iz}\sum_{n=-\infty}^{\infty} q^{(n+1)^2}e^{2(n+1)iz} = q^{-1}e^{-2iz}\vartheta_3(z;q) \quad \text{(B.10)}$$

We can prove similar expresions for the other theta functions. This is the reason why the theta functions are called quasi-double periodic functions.

The table (B.1) illustrates the quasi-double periodicity of the Jacobi theta functions, where $w = q^{-1}e^{-2iz}$. Using the same argument we can also prove:

ϑ_i	$\vartheta_i(z+\pi)/\vartheta_i(z)$	$\vartheta_i(z+\pi\tau)/\vartheta_i(z)$
ϑ_1	-1	$-w$
ϑ_2	-1	w
ϑ_3	1	w
ϑ_4	1	$-w$

Table B.1: Quasi-double periodicity of the Jacobi theta functions ($w = q^{-1}e^{-2iz}$)

$$\vartheta_3'(z|\tau) = \sum_{n=-\infty}^{\infty} q^{n^2}2ine^{2inz} \quad \text{(B.11)}$$

And then:

$$\vartheta_3'(z+\pi|\tau) = \vartheta_3'(z|\tau) \quad \text{(B.12a)}$$

$$\vartheta_3'(z+\pi\tau|\tau) = \sum_{n=-\infty}^{\infty} q^{n^2}q^{2n}2ine^{2inz} = q^{-1}e^{-2iz}\vartheta_3'(z|\tau) - 2i\vartheta_3(z|\tau) \quad \text{(B.12b)}$$

It is easy to see that the same relation hold for the other three theta functions, so we can write:

$$\frac{\vartheta_i'(z+\pi|\tau)}{\vartheta_i(z+\pi|\tau)} = \frac{\vartheta_i'(z|\tau)}{\vartheta_i(z|\tau)} \quad \text{(B.13a)}$$

$$\frac{\vartheta_i'(z+\pi\tau|\tau)}{\vartheta_i(z+\pi\tau|\tau)} = \frac{\vartheta_i'(z|\tau)}{\vartheta_i(z|\tau)} - 2i \quad \text{(B.13b)}$$

Now we can investigate the zeros of the theta functions. Let $\Box$ represent a cell of corners $t, t+\pi, t+\pi+\tau\pi, t+\tau\pi$.

Theorem B.2.1 *$\vartheta_i(z|\tau)$ has one and only one zero in $\Box$.*

Proof B.2.1 *Since $\vartheta_i(z|\tau)$ is analytic the number of zeros of $\vartheta_i(z|\tau)$ is:*

$$\frac{1}{2\pi i}\int_{\Box}\frac{\vartheta_i'(z|\tau)}{\vartheta_i(z|\tau)}\mathrm{d}z \tag{B.14}$$

Dividing the integral in four as in (B.2), we have:

$$\frac{1}{2\pi i}\int_t^{t+\pi}\left\{\frac{\vartheta_i'(z|\tau)}{\vartheta_i(z|\tau)}-\frac{\vartheta_i'(z+\pi\tau|\tau)}{\vartheta_i(z+\pi\tau|\tau)}\right\}\mathrm{d}z-\frac{1}{2\pi i}\int_t^{t+\pi\tau}\left\{\frac{\vartheta_i'(z|\tau)}{\vartheta_i(z|\tau)}-\frac{\vartheta_i'(z+\pi|\tau)}{\vartheta_i(z+\pi|\tau)}\right\}\mathrm{d}z \tag{B.15}$$

That by equations (B.13) is one.

Now we can prove a very important theorem for us, about some kind of "uniqness" of the theta functions

Theorem B.2.2 *If $f(z)$ is an analytic function with the same quasi-double periodicity of $\vartheta_i(z|\tau)$, then $f(z)=A\vartheta_i(z|\tau)$*

Proof B.2.2 *Let us define $\psi(z)$*

$$\psi(z)=\frac{f(z)}{\vartheta_i(z|\tau)} \tag{B.16}$$

Then $\psi(z)$ is a double periodic function that has at most order one (the zero of ϑ_i). Then it must be a constant A, so we have:

$$\psi(z)=A\vartheta_i(z|\tau) \tag{B.17}$$

B.2.3 A relation between theta functions of zero argument.

In this section we will just mention a relation between the theta functions of zero argument that will be necessary for us in the next section.

Theorem B.2.3 $\vartheta_1'(0|\tau)=\vartheta_2(0|\tau)\vartheta_3(0|\tau)\vartheta_4(0|\tau)$

Proof B.2.3 *Several proofs of this proposition has been given but any of them are simple. A proof based on the expresions of the theta functions as products can be found in [6].*

B.2.4 Jacobi's Imaginary transformation

Here we will prove an interesting relation between $\vartheta_i(z|\tau)$ and $\vartheta_i(\frac{z}{\tau}|-\frac{1}{\tau})$. First of all, let us define:

$$\psi(z)=\exp\left(\frac{z^2}{\pi i\tau}\right)\frac{\vartheta_3(\frac{z}{\tau}|-\frac{1}{\tau})}{\vartheta_3(z|\tau)} \tag{B.18}$$

Now the periodicity conditions for $\psi(z)$ is:

$$\psi(z+\pi\tau) = \psi(z) \tag{B.19a}$$
$$\psi(z+\pi) = \psi(z) \tag{B.19b}$$

So $\psi(z)$ is an elliptic function of, at most, order one. So $\psi(z) = A = \text{cte}$.

$$A\vartheta_3(z|\tau) = \exp\left(\frac{z^2}{\pi i\tau}\right)\vartheta_3(\frac{z}{\tau}| - \frac{1}{\tau}) \tag{B.20a}$$

Writing in this expression $z + \frac{\pi}{2}, z + \frac{\pi\tau}{2}, z + \frac{\pi}{2} + \frac{\pi\tau}{2}$ in turn of z we get:

$$\begin{aligned} A\vartheta_1(z|\tau) &= -i\exp\left(\frac{z^2}{\pi i\tau}\right)\vartheta_1(\frac{z}{\tau}| - \frac{1}{\tau}) && \text{(B.20b)} \\ A\vartheta_2(z|\tau) &= \exp\left(\frac{z^2}{\pi i\tau}\right)\vartheta_4(\frac{z}{\tau}| - \frac{1}{\tau}) && \text{(B.20c)} \\ A\vartheta_4(z|\tau) &= \exp\left(\frac{z^2}{\pi i\tau}\right)\vartheta_2(\frac{z}{\tau}| - \frac{1}{\tau}) && \text{(B.20d)} \end{aligned}$$

To determine the value of A, we just differenciate equation (B.20b) and put $z = 0$, to obtain:

$$A\vartheta_1'(0|\tau) = \frac{1}{-i\tau}\vartheta_1'(0| - \frac{1}{\tau}) \tag{B.21}$$

But

$$\begin{aligned} \vartheta_1'(0|\tau) &= \vartheta_2(0|\tau)\vartheta_3(0|\tau)\vartheta_4(0|\tau) && \text{(B.22a)} \\ \vartheta_1'(0| - \frac{1}{\tau}) &= \vartheta_2(0| - \frac{1}{\tau})\vartheta_3(0| - \frac{1}{\tau})\vartheta_4(0| - \frac{1}{\tau}) && \text{(B.22b)} \end{aligned}$$

Dividing this two equations we get:

$$A = \pm\sqrt{-i\tau} \tag{B.23}$$

To determine the correct sign we observe that $A\vartheta_3(z|\tau) = \vartheta_3(z| - \frac{1}{\tau})$ and when τ is a pure imaginary both functions are positive, so we must choose the + sign. Finally we obtain:

$$\begin{aligned} \vartheta_3(z|\tau) &= (-i\tau)^{-1/2}\exp\left(\frac{z^2}{\pi i\tau}\right)\vartheta_3(\frac{z}{\tau}| - \frac{1}{\tau}) && \text{(B.24a)} \\ \vartheta_1(z|\tau) &= -i(-i\tau)^{-1/2}\exp\left(\frac{z^2}{\pi i\tau}\right)\vartheta_1(\frac{z}{\tau}| - \frac{1}{\tau}) && \text{(B.24b)} \\ \vartheta_2(z|\tau) &= (-i\tau)^{-1/2}\exp\left(\frac{z^2}{\pi i\tau}\right)\vartheta_4(\frac{z}{\tau}| - \frac{1}{\tau}) && \text{(B.24c)} \\ \vartheta_4(z|\tau) &= (-i\tau)^{-1/2}\exp\left(\frac{z^2}{\pi i\tau}\right)\vartheta_2(\frac{z}{\tau}| - \frac{1}{\tau}) && \text{(B.24d)} \end{aligned}$$

We can use this relations for the special case $\tau = ia$, to obtain a result used in the text:

$$\vartheta_3(z|ia) = \frac{1}{\sqrt{a}}e^{-\frac{z^2}{\pi a}}\vartheta_3\left(\frac{z}{ia}|\frac{i}{a}\right) = \frac{1}{\sqrt{a}}\sum_{n=-\infty}^{\infty} e^{-\frac{\pi}{a}(n-\frac{z}{\pi})^2} \tag{B.25}$$

B.3 Some relations between Jacobi Theta functions

In this section we are going to demonstrate some useful relations between different Theta functions.

First of all we are going to sum two theta functions, but one displaced respect the other:

$$\vartheta_3(z|\tau) + \vartheta_3(z+a|\tau) = \sum_n e^{i\pi n^2\tau} e^{2inz}[1 + e^{2ina}] \tag{B.26}$$

if we choose $a = \pi/2$, then we get

$$\sum_n e^{i\pi n^2\tau} e^{2inz}[1 + (-)^n] \tag{B.27}$$

we can rewrite this as a sum only over the odd terms. We make $n = 2k$, and then we obtain

$$2\sum_k e^{4i\pi k^2\tau} e^{4ikz} = 2\vartheta_3(2z|4\tau) \tag{B.28}$$

So we obtain the useful relation

$$\vartheta_3(2z|4\tau) = \frac{1}{2}\left[\vartheta_3(z|\tau) + \vartheta_3\left(z + \frac{\pi}{2}|\tau\right)\right] = \frac{1}{2}\left[\vartheta_3(z|\tau) + \vartheta_4(z|\tau)\right] \tag{B.29a}$$

If we add two second theta functions, we will get a similar relation:

$$\vartheta_2(2z|4\tau) = \frac{1}{2}\left[\vartheta_3(z|\tau) - \vartheta_3\left(z + \frac{\pi}{2}|\tau\right)\right] = \frac{1}{2}\left[\vartheta_3(z|\tau) - \vartheta_4(z|\tau)\right] \tag{B.29b}$$

Adding and subtracting this two relations, we also get

$$\vartheta_3(z|\tau) = \vartheta_3(2z|4\tau) + \vartheta_2(2z|4\tau) \tag{B.29c}$$

$$\vartheta_4(z|\tau) = \vartheta_3(2z|4\tau) - \vartheta_2(2z|4\tau) \tag{B.29d}$$

Now we will calculate the modulus of the third theta function in the special case where the tau parameter is a pure imaginary. In this case we get $(z = x + iy)$

$$|\vartheta_3(z|\tau)|^2 = \sum_{nm} e^{i\pi(\tau n^2 - \tau^* m^2)} e^{2i(nz - mz^*)} \tag{B.30}$$

we make $n + m = n_1, n - m = n_2$, and, in the case $\tau = ib$, we have

$$|\vartheta_3(z|ib)|^2 = \sum_{n_1 n_2} e^{-\frac{b\pi}{2}(n_1^2 + n_2^2)} e^{ixn_1} e^{-yn_2}[1 + (-)^{n_1+n_2}] \tag{B.31}$$

we can separate this expression to obtain:

$$|\vartheta_3(z|ib)|^2 = \sum_{n_1} e^{-\frac{b\pi}{2}n_1^2} e^{2ixn_1} \sum_{n_2} e^{-\frac{b\pi}{2}n_2^2} e^{-2yn_2} + \sum_{n_1}(-)^{n_1} e^{-\frac{b\pi}{2}n_1^2} e^{2ixn_1} \sum_{n_2}(-)^{n_2} e^{-\frac{b\pi}{2}n_2^2} e^{-2yn_2} \tag{B.32}$$

so we obtain:

$$|\vartheta_3(z|ib)|^2 = \frac{1}{2}\left[\vartheta_3(x|ib/2)\vartheta_3(iy|ib/2) + \vartheta_4(x|ib/2)\vartheta_4(iy|ib/2)\right] \tag{B.33}$$

Now we can use the Jacobi imaginary transformation for the Theta functions of pure imaginary argument, to obtain

$$|\vartheta_3(z|ib)|^2 = \sqrt{\frac{1}{2b}} e^{2y^2/(\pi b)}\left[\vartheta_3(x|ib/2)\vartheta_3(2y/b|i2/b) + \vartheta_4(x|ib/2)\vartheta_2(2y/b|i2/b)\right] \tag{B.34}$$

And finally using the previous relations between the Theta functions of double argument, we obtain

$$|\vartheta_3(z|ib)|^2 = \sqrt{\tfrac{1}{8b}}e^{2y^2/(\pi b)}\,[\;\begin{aligned}&\vartheta_3(x|ib/2)\vartheta_3(y/b|i/2b)+\\&\vartheta_3(x|ib/2)\vartheta_4(y/b|i/2b)+\\&\vartheta_4(x|ib/2)\vartheta_3(y/b|i/2b)-\\&\vartheta_4(x|ib/2)\vartheta_4(y/b|i/2b)\;]\end{aligned} \tag{B.35}$$

B.4 Theta functions with characteristic

Definition B.4.1 *We define the Theta functions with characteristics* $\vartheta\left[\begin{array}{c}a\\b\end{array}\right](z|\tau)$ *as*

$$\vartheta\left[\begin{array}{c}a\\b\end{array}\right](z|\tau) = \sum_{n\in\mathbb{Z}} e^{i\pi\tau(n+a)^2}e^{2i(n+a)(z+\pi b)} \tag{B.36}$$

With this definition, the theta functions with characteristics are a generalisation of the four classical Jacobi theta functions in the sense that

$$\vartheta\left[\begin{array}{c}1/2\\1/2\end{array}\right](z|\tau) = -\vartheta_1(z|\tau) \tag{B.37a}$$

$$\vartheta\left[\begin{array}{c}1/2\\0\end{array}\right](z|\tau) = \vartheta_2(z|\tau) \tag{B.37b}$$

$$\vartheta\left[\begin{array}{c}0\\0\end{array}\right](z|\tau) = \vartheta_3(z|\tau) \tag{B.37c}$$

$$\vartheta\left[\begin{array}{c}0\\1/2\end{array}\right](z|\tau) = \vartheta_4(z|\tau) \tag{B.37d}$$

this theta functions with characteristic has the following quasi-periodicity properties

$$\vartheta\left[\begin{array}{c}a\\b\end{array}\right](z+\pi|\tau) = e^{2\pi ia}\vartheta\left[\begin{array}{c}a\\b\end{array}\right](z|\tau) \tag{B.38a}$$

$$\vartheta\left[\begin{array}{c}a\\b\end{array}\right](z+\pi\tau|\tau) = e^{-i\pi\tau}e^{-2i(z+\pi b)}\vartheta\left[\begin{array}{c}a\\b\end{array}\right](z|\tau) \tag{B.38b}$$

and the dual translations in the characteristic a are given by

$$\vartheta\left[\begin{array}{c}a+1\\b\end{array}\right](z|\tau) = \vartheta\left[\begin{array}{c}a\\b\end{array}\right](z|\tau) \tag{B.39a}$$

$$\vartheta\left[\begin{array}{c}a-\frac{1}{\tau}\\b\end{array}\right](z|\tau) = e^{i\pi/\tau}e^{-2i(z+b\pi)/\tau}\vartheta\left[\begin{array}{c}a\\b\end{array}\right](z|\tau) \tag{B.39b}$$

the theta functions with characteristic can be expressed in terms of the classical third Jacobi theta function

$$\vartheta\left[\begin{array}{c}a\\b\end{array}\right](z|\tau) = e^{i\pi\tau a^2}e^{2ia(z+b\pi)}\vartheta_3(z+\pi a\tau+\pi b|\tau) \tag{B.40}$$

this means that the theta functions with characteristic has also one and only one zero in each cell $\square$

$$\vartheta\left[\begin{array}{c}a\\b\end{array}\right](z_c|\tau) = 0 \Longrightarrow z_c = \pi\left[\frac{1}{2} - b + \tau\left(\frac{1}{2} - a\right)\right] \tag{B.41}$$

Now we will analyse the products of theta functions, and we will prove some interesting theorems.

Theorem B.4.1 *Let f(z) be an analytic function without poles, with the same boundary conditions as* $\vartheta_3^q(z|\tau)$*, and with (at least)* $q-1$ *zeros in any fundamental cell* $\Box$ *situated in* $c_1, \ldots, c_{q-1}$*. Then* $f(z)$ *has exactly* q *zeros, and*

$$f(z) = A\vartheta_3(z-a_1|\tau)\cdots\vartheta_3(z-a_{q-1}|\tau)\vartheta_3(z-a_q|\tau) \tag{B.42}$$

where $a_i = c_i - \frac{\pi}{2}(1+\tau)$ *for* $i = 1, \ldots, q-1$*, and* $a_q = -\sum_{i=1}^{q-1} a_i$*, and* A *is a constant.*

Proof B.4.1 *First we will prove that* $\vartheta_3(z-a_1|\tau)\cdots\vartheta_3(z-a_{q-1}|\tau)\vartheta_3(z-a_q|\tau)$ *have the same boundary conditions that* $\vartheta_3^q(z|\tau)$*, but this follows easily from the quasi periodicity conditions of the table (B.1) if* $\sum_{i=1}^{q} a_i = 0$

Now we can define

$$\psi(z) = \frac{f(z)}{\vartheta_3(z-a_1|\tau)\cdots\vartheta_3(z-a_{q-1}|\tau)\vartheta_3(z-a_q|\tau)} \tag{B.43}$$

This is an elliptic function with at most a simple pole situated in $z = \frac{\pi}{2}(1+\tau) + \sum_{i=1}^{q-1} a_i$ *in the cell* $\Box$*, so it must be a constant* $\psi(z) = A$*. On the other hand* $\vartheta_3(z-a_1|\tau)\cdots\vartheta_3(z-a_q|\tau)$*, has exactly* q *zeros in each cell situated in* $c_i = a_i + \frac{\pi}{2}(1+\tau)$*, then these must be also the zeros of* $f(z)$*.*

C

The Hilbert space of quasi periodic functions

Here we will study fields living in the two dimensional Torus. The coordinates of the Torus will be labelled (x_1, x_2), and the orthogonal periods $\hat{l}_1 = (l_1, 0), \hat{l}_2 = (0, l_2)$, where l_i are the Torus sizes. Then the Torus area is given by $\mathcal{A} = l_1 l_2$, and we will label the first Chern class with q. This fields will obey the following boundary conditions

$$\phi(x + \hat{l}_1) = e^{\imath\pi q \frac{x_2}{l_2}} \phi(x) \tag{C.1a}$$

$$\phi(x + \hat{l}_2) = e^{-\imath\pi q \frac{x_1}{l_1}} \phi(x) \tag{C.1b}$$

and q is the flux number (first Chern class). These fields with the scalar product

$$\langle\phi|\psi\rangle = \frac{1}{\mathcal{A}} \int_{\mathbb{T}^2} d^2x\, \overline{\phi}\psi \tag{C.2}$$

makes a pre-Hilbert space $\mathcal{H}_q$. We want to study the structure of this pre-Hilbert space.

This appendix is self contained except that some results depends on the definitions and properties of the Theta functions described in the appendix B. This appendix is based in the appendix of [71] and the material of [34].

C.1 A basis for our space

In order to find an orthonormal basis for our pre-Hilbert space, we will look for a complete set of commuting operators, and we will use its eigenvectors as our basis. But first of all, we will write our fields in a special way.

Theorem C.1.1 *Every element $|\,\phi\rangle \in \mathcal{H}_q$ can be written as*

$$|\,\phi\rangle = e^{\imath \frac{f}{2} x_1 x_2} \sum_k h_s \left(x_2 + \frac{k l_2}{q} \right) e^{2\pi\imath k \frac{x_1}{l_1}} \tag{C.3}$$

where $s = k \bmod q$, $f = 2\pi q/\mathcal{A}$ and h_s are q arbitrary functions.

Proof C.1.1 *If we define $\eta(x)$ as (note that $\eta(x) \notin \mathcal{H}_q$)*

$$\eta(x) = e^{-\imath \frac{f}{2} x_1 x_2} \phi(x) \tag{C.4}$$

the boundary conditions for $\eta(x)$ *are*

$$\eta(x+\hat{l}_1)=\eta(x) \tag{C.5a}$$

$$\eta(x+\hat{l}_2)=e^{-\imath f l_2 x_1}\eta(x) \tag{C.5b}$$

This means, first that we can Fourier expand in the variable x_1

$$\eta(x)=\sum_k h_k(x_2)e^{2\pi\imath\frac{x_1}{l_1}} \tag{C.6}$$

second, that the "modes" $h_k(x_2)$ *must obey the constraint*

$$h_{k+q}(x_2)=h_k(x_2+l_2) \tag{C.7}$$

that is solved writing

$$h_k(x_2)=h_s(x_2+kl_2/q) \tag{C.8}$$

where $s=k \bmod q$*, that is the desired result.*

So any element of $\mathcal{H}_q$ is given by choosing q arbitrary functions $h_s(u)$, the boundary conditions gives the rest.

Based of the physical problems that we will solve, we are interested in how the covariant derivatives with a constant background field acts over these elements. Note that an usual derivative is not a well defined operator in our Hilbert space, since it changes the boundary conditions of the fields over it acts, but a covariant derivative with a gauge connection that transform under the same transition functions is a well defined operator in our Hilbert space. So we are interested in the operators

$$D_i^{(0)}:\mathcal{H}_q\longrightarrow\mathcal{H}_q \tag{C.9}$$

where $D_i^{(0)}=\partial_i-iA_i^{(0)}$ and

$$A_i^{(0)}=-\varepsilon_{ij}f\frac{x_j}{2} \tag{C.10}$$

These operators obey (up to a normalisation) the Hisenberg type commutation relations

$$[D_1^{(0)},D_2^{(0)}]=-if \tag{C.11}$$

so we can define annihilation and creation operators

$$a=\frac{-\imath}{\sqrt{2f}}\left(D_1^{(0)}+\imath D_2^{(0)}\right)\equiv\frac{-\imath}{\sqrt{2f}}D_-^{(0)} \tag{C.12a}$$

$$a^+=\frac{-\imath}{\sqrt{2f}}\left(D_1^{(0)}-\imath D_2^{(0)}\right)\equiv\frac{-\imath}{\sqrt{2f}}D_+^{(0)} \tag{C.12b}$$

that have the usual commutation relations

$$[a,a^+]=1 \tag{C.13}$$

Once this is noted, we know form the standard quantum mechanical formulation of the harmonic oscillator the relevant quantities. If we define the self-adjoint operator (usually called number operator)

$$N=a^+a \tag{C.14}$$

we know that its eigenfunctions are labelled by a non-negative integer number n

$$N \mid n\rangle = n \mid n\rangle \tag{C.15}$$

and the annihilation-creation operator act on these basis elements as follow

$$\begin{aligned} &a \mid n\rangle = \sqrt{n} \mid n-1\rangle \quad (n>0) \\ &a \mid 0\rangle = 0 \end{aligned} \tag{C.16a}$$

$$a^{+} \mid n\rangle = \sqrt{n+1} \mid n+1\rangle \tag{C.16b}$$

The state $\mid 0\rangle$ is usually called *ground state*. We can also study how this annihilation-creation operators acts over an arbitrary element of our Hilbert space.

$$a \mid \phi\rangle = e^{\imath \frac{f}{2} x_1 x_2} \sum_k \frac{1}{\sqrt{2}} \left(h_s'(u) + u h_s(u)\right) e^{2\pi \imath k \frac{x_1}{l_1}} \tag{C.17a}$$

$$a^{+} \mid \phi\rangle = e^{\imath \frac{f}{2} x_1 x_2} \sum_k \frac{1}{\sqrt{2}} \left(-h_s'(u) + u h_s(u)\right) e^{2\pi \imath k \frac{x_1}{l_1}} \tag{C.17b}$$

where $u = \sqrt{f}\,(x_2 + kl_2/q)$ and $s = k \bmod q$. It is also easy to compute the action of the number operator over an element of our Hilbert space

$$a^{+} a \mid \phi\rangle = e^{\imath \frac{f}{2} x_1 x_2} \sum_k -\frac{1}{2} \left[h_s''(u) + (1-u^2) h_s(u)\right] e^{2\pi \imath k \frac{x_1}{l_1}} \tag{C.18}$$

now the eigenvalue equation

$$a^{+} a \mid \phi\rangle = n \mid \phi\rangle \tag{C.19}$$

transforms in the following equation for $h_s(u)$

$$h_s''(u) + (2n+1-u^2) h_s(u) = 0 \tag{C.20}$$

That is the equation for the quantum harmonic oscillator, and has two solutions, the trivial solution $h_s(u) = 0$ and

$$h_s(u) = e^{-u^2/2} H_n(u) \tag{C.21}$$

where $H_n(u)$ is the Hermite Polynomial. That means that the q functions that we have to define to give an state of our Hilbert space, if we want it to be an eigenvalue of the number operator, must be either zero or the function given in equation (C.21).

To make the basis complete, one simple solution is to choose all the h_s zero but one. We have q possibilities to do that for each value of n, and we will label them $\mid n, s\rangle$, all of them having the same eigenvalue of the number operator. This will give us our desired basis

Definition C.1.1 *A (convenient) basis for our Hilbert space is given by the functions* $\mid n, s\rangle \in \mathcal{H}_q$

$$\mid n, s\rangle = \left(\frac{2ql_2}{l_1}\right)^{1/4} e^{\imath \frac{f}{2} x_1 x_2} \sum_{k \in s + q\mathbb{Z}} e^{-u^2/2} H_n(u) e^{2\pi \imath k \frac{x_1}{l_1}} \qquad n = 0, \ldots \infty; s = 1, \ldots, q \tag{C.22}$$

where $u = \sqrt{f}\,(x_2 + kl_2/q)$*. This basis is orthonormal in the usual sense*

$$\langle n, s | n', s'\rangle = \delta_{nn'} \delta_{ss'} \tag{C.23}$$

Our creation and annihilation operators act over the elements of this basis in the usual way

$$\begin{aligned} a \mid n,s\rangle &= \sqrt{n} \mid n-1,s\rangle \quad (n>0) \\ a \mid 0,s\rangle &= 0 \end{aligned} \tag{C.24a}$$

$$a^+ \mid n,s\rangle = \sqrt{n+1} \mid n+1,s\rangle \tag{C.24b}$$

and they are eigenfunctions of the number operator $N = a^+a$

$$a^+a \mid n,s\rangle = n \mid n,s\rangle \tag{C.25}$$

It is important to note here that the "ground state" has a simple expression in terms of the theta functions with characteristics defined in the appendix B

$$\mid 0,s\rangle = \left(\frac{2ql_2}{l_1}\right)^{1/4} e^{\imath\frac{f}{2}x_1x_2} e^{-\frac{f}{2}x_2^2} \vartheta \left[\begin{array}{c} s/q \\ 0 \end{array}\right] \left(\frac{2\pi qz}{l_1} | \imath q\frac{l_2}{l_1}\right) \tag{C.26}$$

where $z = (x_1 + \imath x_2)/2$. For those that are not happy enough with this definition of the basis "by hand" a more detailed explanation with some important properties of the elements of the basis will follow.

C.2 Properties of the elements of the basis

A normal translation will map functions of our Hilbert space out of it, since they will change the boundary conditions, this is the reason why the usual derivatives, that are the generators of the translations, are not well defined operators of our Hilbert space. But if we use the covariant derivatives instead, they will generate some kind of "translations".

$$\mathcal{T}_1(a)\phi(x) = e^{aD_1^{(0)}}\phi(x) = e^{ia\frac{f}{2}l_1x_2}\phi(x+(a,0)) \tag{C.27a}$$

$$\mathcal{T}_2(a)\phi(x) = e^{aD_2^{(0)}}\phi(x) = e^{-ia\frac{f}{2}l_2x_1}\phi(x+(0,a)) \tag{C.27b}$$

These operators are a translation mixed with a gauge transformation. These operators are well defined operators in our Hilbert space

$$\mathcal{T}_i(a) : \mathcal{H}_q \longrightarrow \mathcal{H}_q \tag{C.28}$$

These operators does not commute, for the same reason as the covariant derivatives do not, and their commutator is related with the first Chern class

$$\mathcal{T}_1(a)\mathcal{T}_2(b) = e^{-\imath fab}\mathcal{T}_2(b)\mathcal{T}_1(a) \tag{C.29}$$

Also this operators doesnt commute in general with the "number" operator defined above. A general gauge transformation will also map the fields out of our Hilbert space, but there is a subset of them that define operators in our Hilbert space

$$\mathcal{U}_i(k)\phi(x) = e^{2\pi\imath k\frac{x_i}{l_i}}\phi(x) \tag{C.30}$$

with $k \in \mathbb{Z}$. This operators commute between them, and with its help we can define two operators that commute with the "number" operator

$$\mathcal{K}^{(1)} = \mathcal{U}_1(1)\mathcal{T}_2(l_2/q) \tag{C.31a}$$

$$\mathcal{K}^{(2)} = \mathcal{U}_2(1)\mathcal{T}_1(-l_1/q) \tag{C.31b}$$

It is easy to prove that the operators $\mathcal{K}^{(i)}$ commute with the number operator and have the property

$$\left(\mathcal{K}^{(i)}\right)^q = \mathbb{I} \tag{C.32}$$

and so their eigenvalues must be of the form $e^{-2\pi\imath l/q}$. So we can choose a basis of eigenstates both of the number operator and one of the operators $\mathcal{K}^{(i)}$ (note that the two operators $\mathcal{K}^{(i)}$ do not commute between them). We will choose $\mathcal{K}^{(1)}$ for that purpose, and it is easy to show, that any $\mid \phi\rangle \in \mathcal{H}_q$

$$\mid \phi\rangle = e^{\imath \frac{f}{2} x_1 x_2} \sum_k h_s \left(x_2 + \frac{kl_2}{q}\right) e^{2\pi\imath k \frac{x_1}{l_1}} \tag{C.33}$$

is an eigenfunction of $\mathcal{K}^{(1)}$ with eigenvalue $e^{-2\pi\imath l/q}$ if and only if all the h_s are zero but h_l. This means that we can say that our basis $\{\mid n, s\rangle\}$ is a basis of common eigenfunctions of both the number operator and the operator $\mathcal{K}^{(1)}$ with eigenvalues

$$N \mid n, s\rangle = n \mid n, s\rangle \tag{C.34a}$$

$$\mathcal{K}^{(1)} \mid n, s\rangle = e^{-2\pi\imath s/q} \mid n, s\rangle \tag{C.34b}$$

and that $\mathcal{K}^{(1)}$ generates a $\mathbb{Z}_q$ subgroup under which $\mathcal{H}_q$ decomposes into q orthogonal subspaces

$$\mathcal{H}_q = \oplus_{i=1}^q \mathcal{H}_{q,i} \tag{C.35}$$

We can now compute the matrix elements of the operator $\mathcal{T}_i(a)$

$$\langle n, s \mid \mathcal{T}_i(a) \mid n', s'\rangle = \delta_{ss'} e^{\imath\beta_i(n'-n)} (-)^{m+n'} e^{-z^2/2} z^{|n-n'|} \sum_{j=0}^m \frac{(-)^j \sqrt{n!n'!} z^{2j}}{j!(m-j)!(j+|n-n'|)!} \tag{C.36}$$

where $m = \min(n, n')$, $z = a\sqrt{f/2}$ and $\beta_1 = -\pi/2, \beta_2 = \pi$. Unce we have this, we can compute the matrix elements of the operator $\mathcal{U}_1(k_1)\mathcal{U}_2(k_2)$, since

$$\mathcal{U}_1(k_1)\mathcal{U}_2(k_2) = e^{-\imath\pi\frac{k_1k_2}{q}} \left(\mathcal{K}^{(2)}\right)^{-k1} \left(\mathcal{K}^{(1)}\right)^{k2} \mathcal{T}_1(k_1x_1/l_1)\mathcal{T}_2(k_2x_2/l_2) \tag{C.37}$$

the matrix elements are

$$\begin{aligned} \langle n, s \mid \mathcal{U}_1(k_1)\mathcal{U}_2(k_2) \mid n', s'\rangle &= (-)^{(m+n')} \delta_{s,s'+k_1} e^{2\pi\imath\frac{s'k_2}{q}} e^{-\pi\imath\frac{k_1k_2}{q}} e^{\imath\alpha(n'-n)} \times \\ &\times \; e^{-\xi/2} \xi^{|n-n'|/2} \sum_{j=0}^m \frac{(-)^j \sqrt{n!n'!} \xi^j}{j!(m-j)!(j+|n-n'|)!} \end{aligned} \tag{C.38}$$

where α and ξ can be obtained from the complex number

$$\sqrt{\xi} e^{\imath\alpha} \equiv \sqrt{\frac{\pi\tau}{q}} \left(k_1 + \imath\frac{k_2}{\tau}\right) \tag{C.39}$$

and $\tau = l_2/l_1$. Take into account that the equation (C.38) gives us the Fourier modes of the product of two elements of our basis.

Now let $\mathcal{R}$ the operator that makes a 180º rotation over a field

$$\mathcal{R}\phi(x) = \phi(-x) \tag{C.40}$$

this operator acts over the basis elements in the following way

$$\mathcal{R} \mid n, s\rangle = (-)^n \mid n, q-s\rangle \tag{C.41}$$

D

Operator perturbation theory

> To first approximation, the human brain is a harmonic oscillator.
>
> Barry Simon

In this chapter we will show how to obtain the eigenvalues as a power series in a (small) quantity ϵ of an operator

$$\hat{O} = \sum_n \hat{O}^{(n)} \epsilon^n \tag{D.1}$$

supossed that we know the eigenvalues $\lambda_i^{(0)}$ and eigenfunctions $\mid i\rangle$ of the operator $\hat{O}^{(0)}$.

$$\hat{O}^{(0)} \mid i\rangle = \lambda_i^{(0)} \mid i\rangle \tag{D.2}$$

That is to say, we want to know the eigenvalues

$$\hat{O} \mid \psi_i\rangle = \lambda_i \mid \psi_i\rangle \tag{D.3}$$

as a power series

$$\lambda_i = \sum_n \lambda_i^{(n)} \epsilon^n \tag{D.4}$$

D.1 General definitions

Definition D.1.1 *Given an operator $\hat{O}$ we will call the* resolvent of $\hat{O}$ *to the function of the complex variable z*

$$G(z) = \frac{1}{z - \hat{O}} \tag{D.5}$$

the nice thing about the resolvent of an operator is (as we will show) that is an analitic function of z with simple poles located at the eigenvalues of $\hat{O}$.

Definition D.1.2 *We will define the operator $P_i^{(k)}$ as the projector over the eigenstate $\mid i\rangle$ of the operator $\hat{O}^{(k)}$ and P_i as the projector over the eigenstate $\mid \psi_i\rangle$ of the operator $\hat{O}$. In particular it is clear that*

$$P_i^{(0)} = \mid i\rangle\langle i \mid \tag{D.6}$$

Obviously, these operators obey the rules

$$P_iP_j = \delta_{ij}P_i \tag{D.7a}$$

$$\sum_i P_i = 1 \tag{D.7b}$$

Theorem D.1.1 *The resolvent of any operator $\hat{O}$ can be written as*

$$G(z) = \sum_i \frac{P_i}{z - \lambda_i} \tag{D.8}$$

Proof D.1.1 *First of all it is easy to show that*

$$G(z)P_i = \frac{P_i}{z - \lambda_i} \tag{D.9}$$

since for any state $\mid \psi\rangle$, we can write

$$\mid \psi\rangle = \sum_n c_n \mid \psi_n\rangle \tag{D.10}$$

then

$$P_i \mid \psi\rangle = c_i \mid \psi_i\rangle \tag{D.11}$$

so

$$G(z)P_i \mid \psi\rangle = \frac{c_i \mid \psi_i\rangle}{z - \lambda_i} = \frac{P_i}{z - \lambda_i} \mid \psi\rangle \tag{D.12}$$

Now we only have to sum over the projectors, and use the property (D.7b).

thanks to this theorem it is clear that each eigenvalue of $\hat{O}$ is a simple pole of its resolvent, and that the projectors are the residues, so, if Γ is a path with some eigenvalues inside, we can write

$$P_\Gamma = \frac{1}{2\pi i}\oint_\Gamma dz\ G(z) \tag{D.13}$$

where P_Γ is the sum of the projectors whose eigenvalues are inside Γ. If we multiply (D.13) by $\hat{O}$, we obtain

$$\hat{O}P_\Gamma = \frac{1}{2\pi i}\oint_\Gamma dz\ zG(z) \tag{D.14}$$

D.2 Series expansion

Now we will write the operator $\hat{O}$ as the seriers expansion in the parameter ϵ. Its resolvent will be $G(z)$, and we will also use the resolvent of the operator $\hat{O}^{(0)}$, that we will call $G_0(z)$. The first thing that we need is a power expansion of the resolvent $G(z)$.

Theorem D.2.1 *The power expansion of the resolvent $G(z)$ is given by*

$$G(z) = \sum_{n=0}^{\infty} G_0 \left(\sum_{s=1}^{n} \sum_{\sigma_1(j_1,\ldots,j_s;n)} \hat{O}^{(j_1)}G_0\hat{O}^{(j_2)}\cdots\hat{O}^{(j_s)}G_0 \right)\epsilon^n \tag{D.15}$$

where the simbol

$$\sum_{\sigma_k(j_1,\ldots,j_s;n)} \tag{D.16}$$

means the sum over the numbers $j_1, \ldots, j_n$ *whith the property*

$$\sum_{i=1}^{s} j_i = n; \quad j_i \geq k \tag{D.17}$$

Proof D.2.1 *It is clear that we can write*

$$G(z) = \frac{1}{G_0^{-1}(1 - G_0 V)} = \sum_n G_0 (V G_0)^n \tag{D.18}$$

where

$$V = \sum_{n=1}^{\infty} \hat{O}^{(n)} \epsilon^n \tag{D.19}$$

now we have to group the terms with the same power of epsilon, and reorder the sum, to obtain

$$G = G_0 + G_0 \hat{O}^{(1)} G_0 \epsilon + \left(G_0 \hat{O}^{(2)} G_0 + G_0 \hat{O}^{(1)} G_0 \hat{O}^{(1)} G_0 \right) \epsilon^2 + \ldots \tag{D.20}$$

that is the desired result.

Now we want to compute the series expansion of a projector P_a. We will choose a path Γ_a that includes the points $\lambda_a^{(0)}, \lambda_a^{(0)} + \epsilon\lambda_a^{(1)}, \lambda_a^{(0)} + \epsilon\lambda_a^{(1)} + \epsilon^2\lambda_a^{(2)}, \ldots$, and no other eigenvalue of $\hat{O}$. Clearly if ϵ is small enought this can always be done. The projector is given by

$$P_a = \frac{1}{2\pi i} \oint_{\Gamma_a} dz\ G(z) \tag{D.21}$$

if we insert the power expansion of $G(z)$ in this formula, we obtain

$$P_a = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \left(\sum_{s=1}^{n} \sum_{\sigma_1(j_1,\ldots,j_s;n)} \oint_{\Gamma_a} G_0 \hat{O}^{(j_1)} G_0 \hat{O}^{(j_2)} \cdots \hat{O}^{(j_s)} G_0\ dz \right) \epsilon^n \tag{D.22}$$

so we obtain

$$P_a^{(n)} = \frac{1}{2\pi i} \sum_{s=1}^{n} \sum_{\sigma_1(j_1,\ldots,j_n;n)} \oint_{\Gamma_a} G_0 \hat{O}^{(j_1)} G_0 \hat{O}^{(j_2)} \cdots \hat{O}^{(j_s)} G_0\ dz \tag{D.23}$$

now we have to compute this integral. The only singularity of the integrand is in $\lambda_a^{(0)}$. To compute the residue of one of these functions, we use the theorem (D.1.1), to write

$$G_0(z) = \frac{P_a^{(0)}}{z - \lambda_a^{(0)}} + \sum_k (-)^k Q^{(k)} (z - \lambda_a^{(0)})^k \tag{D.24}$$

where

$$Q^{(n)} = \sum_{i \neq a} \frac{P_i}{(\lambda_a^{(0)} - \lambda_i^{(0)})^{(n+1)}} \tag{D.25}$$

We can write this as

$$G_0(z) = \sum_{k=0} S^{(k)}(z-\lambda_a^{(0)})^{k-1} \tag{D.26}$$

where

$$S^{(k)} = \begin{cases} P_a^{(0)} & \text{if } k=0 \\ Q^{(n)} & \text{if } k>0 \end{cases} \tag{D.27}$$

finally the result of *each* of these integrals is

$$\frac{1}{2\pi i}\oint_{\Gamma_a} G_0\hat{O}^{(j_1)}G_0\hat{O}^{(j_2)}\cdots\hat{O}^{(j_s)}G_0\ dz = \sum_{\sigma_0(r_1,\ldots,r_{s+1};s)} S^{(r_1)}\hat{O}^{(j_1)}\cdots\hat{O}^{(j_s)}S^{(r_{s+1})} \tag{D.28}$$

D.3 Expansion for the eigenvalues

D.3.1 Non degenerate eigenvalues

Now we want a power expansion for the eigenvalues. We want to compute

$$\hat{O}P_a = \lambda_a P_a \tag{D.29}$$

by equation (D.14) we have

$$(\hat{O}-\lambda_a^{(0)})P_a = \oint_{\Gamma_a}(z-\lambda_a^{(0)})G(z)\ dz \tag{D.30}$$

using the same expansion as above, we have

$$(\hat{O}-\lambda_a^{(0)})P_a = \sum_{n=1}^{\infty}\left(\sum_{s=1}^{n}\sum_{\sigma_1(j_1,\ldots,j_s;n)}\sum_{\sigma_0(r_1,\ldots,r_{s+1};s-1)} S^{(r_1)}\hat{O}^{(j_1)}S^{(r_1)}\cdots O^{(j_s)}S^{(r_{s+1})}\right)\epsilon^n \tag{D.31}$$

since $tr(P_a)=1$, we have

$$tr\{(\hat{O}-\lambda_a^{(0)})P_a\} = \sum_{n=1}^{\infty}\lambda_a^{(n)}\epsilon^n \tag{D.32}$$

so we have

$$\lambda_a^{(n)} = \sum_{s=1}^{n}\sum_{\sigma_1(j_1,\ldots,j_s;n)}\sum_{\sigma_0(r_1,\ldots,r_{s+1};s-1)} tr\left\{S^{(r_1)}\hat{O}^{(j_1)}S^{(r_1)}\cdots O^{(j_s)}S^{(r_{s+1})}\right\} \tag{D.33}$$

since at least one of the r_i must be zero, using the ciclic property of the trace, we can make this $S^{(0)} = P_a^{(0)} = \mid a\rangle\langle a\mid$ appears to the right, so that we obtain the value of the contribution to the eigenvalue as a sandwich between the state $\mid a\rangle$. The first terms of this expansion gives

$$\lambda_a^{(1)} = tr\left\{S^{(0)}\hat{O}^{(1)}S^{(0)}\right\} = \sum_i \langle i|a\rangle\langle a\mid\hat{O}^{(1)}\mid a\rangle\langle a|i\rangle = \langle a\mid\hat{O}^{(1)}\mid a\rangle \tag{D.34a}$$

$$\begin{aligned}\lambda_a^{(2)} &= tr\Big\{S^{(0)}\hat{O}^{(2)}S^{(0)}+ \\ &S^{(1)}\hat{O}^{(1)}S^{(0)}\hat{O}^{(1)}S^{(0)} + S^{(0)}\hat{O}^{(1)}S^{(1)}\hat{O}^{(1)}S^{(0)} + S^{(0)}\hat{O}^{(1)}S^{(0)}\hat{O}^{(1)}S^{(1)}\Big\} = \\ &\langle a\mid\hat{O}^{(2)}\mid a\rangle + \langle a\mid\hat{O}^{(1)}\textstyle\sum_{i\neq a}\frac{|i\rangle\langle i|}{\lambda_a^{(0)}-\lambda_i^{(0)}}\hat{O}^{(1)}\mid a\rangle\end{aligned} \tag{D.34b}$$

$$\begin{aligned}\lambda_a^{(3)} = \; & \langle a \mid \hat{O}^{(3)} \mid a\rangle + \langle a \mid \hat{O}^{(1)} \textstyle\sum_{i\neq a} \frac{|i\rangle\langle i|}{\lambda_a^{(0)}-\lambda_i^{(0)}} \hat{O}^{(2)} \mid a\rangle + \langle a \mid \hat{O}^{(2)} \textstyle\sum_{i\neq a} \frac{|i\rangle\langle i|}{\lambda_a^{(0)}-\lambda_i^{(0)}} \hat{O}^{(1)} \mid a\rangle \\ & +2\langle a \mid \hat{O}^{(1)} \mid a\rangle\langle a \mid \hat{O}^{(1)} \textstyle\sum_{i\neq a} \frac{|i\rangle\langle i|}{(\lambda_a^{(0)}-\lambda_i^{(0)})^2} \hat{O}^{(1)} \mid a\rangle\end{aligned} \tag{D.34c}$$

D.3.2 Degenerate eigenvalues

Now we will face the most general case: We will call the eigenvalues of $\hat{O}^{(0)}$ simply $\mid nls\rangle$

$$\hat{O}^{(0)} \mid nls\rangle = \lambda_n^{(0)} \mid nls\rangle \tag{D.35}$$

and the eigenvalues of the operator $\hat{O}$, simply $\mid \psi_{nls}\rangle$

$$\hat{O} \mid \psi_{nls}\rangle = \lambda_{nl} \mid \psi_{nls}\rangle \tag{D.36}$$

this is the most general case, in which the operator $\hat{O}^{(0)}$ is invariant under a group of transformations G_0, that makes that the eigenvalues only depend on the label n, and not on l and s. Later, the full operator $\hat{O}$ is invariant only under a subgroup of transformations $G < G_0$. These makes that the eigenvalues of the operator $\hat{O}$ only depend on the labels n, l, and not on s.

First, from (D.14), it is clear that

$$\left(\hat{O} - C\right)^k P_\Gamma = \frac{1}{2\pi i} \oint_\Gamma dz \; (z - C)^k G(z) \tag{D.37}$$

now we will choose the path Γ, such that the eigenvalue $\lambda_n^{(0)}$, and $\lambda_{nl}, (\forall l)$, are inside Γ (see Fig (D.1)).

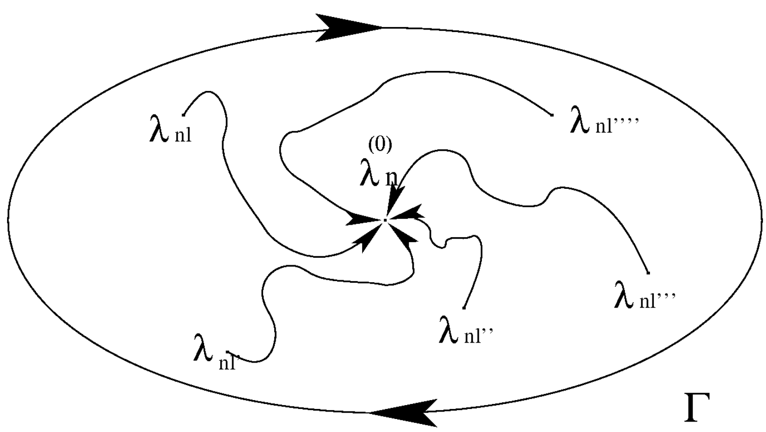

Figure D.1: Contour choosed to compute the eigenvalues in the degenerate case.

Using the perturbative expansion of $G(z)$, we can write

$$(\hat{O}-\lambda_a^{(0)})^k P_\Gamma = \sum_{n=1}^{\infty} \left(\sum_{s=1}^{n} \sum_{\sigma_1(j_1,\ldots,j_s;n)} \sum_{\sigma_0(r_1,\ldots,r_{s+1};s-k)} S^{(r_1)}\hat{O}^{(j_1)}S^{(r_1)}\cdots O^{(j_s)}S^{(r_{s+1})} \right) \epsilon^n \tag{D.38}$$

where, now

$$S^{(n)} = \begin{cases} \sum_{ls} P^{(0)}_{nls} & \text{if } n = 0 \\ \sum_{i\neq n,ls} \frac{P^{(0)}_{ils}}{(\lambda^{(0)}_i - \lambda^{(0)}_n)^{n+1}} & \text{if } n > 0 \end{cases} \tag{D.39}$$

on the other hand, we can compute

$$tr\left[\left(\hat{O} - \lambda^{(0)}_n\right)^k P_\Gamma\right] = d_s \sum_l \left(\lambda_{nl} - \lambda^{(0)}_n\right)^k \tag{D.40}$$

where d_s is the number of eigenvectors with egenvalue λ_{nl}. This equation gives a series of non linear equations, that can be solved in powers of ϵ to obtain the eigenvalues of the operator $\hat{O}$. We can write

$$tr\left[\left(\hat{O} - \lambda^{(0)}_n\right)^k P_\Gamma\right] = d_s \sum_l \sum_i \left(\sum_{\sigma_0(i_1,\ldots,i_k;i)} \lambda^{(i_1)}_{nl} \cdots \lambda^{(i_k)}_{nl} \right) \epsilon^i \tag{D.41}$$

so the equations to solve are $(\forall k)$

$$d_s \sum_l \left(\sum_{\sigma_0(i_1,\ldots,i_k;i)} \lambda^{(i_1)}_{nl} \cdots \lambda^{(i_k)}_{nl} \right) = \sum_{s=1}^{i} \sum_{\sigma_1(j_1,\ldots,j_s;i)} \sum_{\sigma_0(r_1,\ldots,r_{s+1};s-k)} S^{(r_1)} \hat{O}^{(j_1)} S^{(r_1)} \cdots O^{(j_s)} S^{(r_{s+1})} \tag{D.42}$$

E

Code

> Computers help us to solve problems we never had before they came along.
>
> Anonymous

In this appendix, a brief description on how to use the code used along this work to obtain the numerical results is detailed. Almost all the code is written in standard `FORTRAN 90`, and make use of the library `afnl` (a free software `FORTRAN 90` library). The source code, and manual of this library can be found in the homepage of it's `sourceforge` project.

`http://sourceforge.net/projects/afnl`

All the software described below is `free software`, distributed under the terms of the `GPL`[1] license. This means, among other things that your are allowed to run the software, see the source code, copy it any number of times that you want, and even improve the software, and make this improvements available to the public. For more details take a look at the full license. You can download all the source code of the detailed software at

`http://lattice.ft.uam.es/perpag/alberto/codigo_en.php`

E.1 `omega`

E.1.1 Description

This code computes the sums

$$\mathcal{S}(n,n',\vec{k}) = \xi^{|n-n'|/2}\sum_{j=0}^{m}\frac{(-)^j\sqrt{n!n'!}\xi^j}{j!(m-j)!(j+|n-n'|)!} \tag{E.1}$$

where $m = \min(n,n')$, with arbitrary precision float point arithmetics. This code uses the `GNU gmp` library to perform arbitrary precision arithmetics.

[1] The full text of the license can be found at `http://www.gnu.org/copyleft/gpl.html`

E.1.2 Arguments

`n`: Minimum value of n and n'. Default 0.

`N`: Maximum value of n and n'. Default 100.

`k`: Minimum value of k_1 and k_2. Default 0.

`K`: Maximum value of k_1 and k_2. Default 10.

`t`: Value of the aspect ratio l_2/l_1. Default 1.0

`q`: The first chern class q (flux number). Default 1.

`p`: The precision. Default 1000.

E.1.3 Example

Using the command

```
omega.out -n 40 -N 42 -k 0 -K 1 -t 1.0 -q 1
```

we obtain the values $\mathcal{S}(n, n', \vec{k})$ for $40 \leq n, n' \leq 42$ and $0 \leq k_1, k_2 \leq 1$ in the following way:

```
40   40    0    0 1.00000000000000000000e+00
40   40    0    1 -7.76813472064891800324e-01
40   40    1    1 2.88402447666946977930e+00
40   41    0    0 0.00000000000000000000e+00
40   41    0    1 1.08333098260181482577e-01
40   41    1    1 -9.57642800312668658046e-01
40   42    0    0 0.00000000000000000000e+00
40   42    0    1 8.06569686550008052010e-01
40   42    1    1 -3.27883288957379882145e+00
41   41    0    0 1.00000000000000000000e+00
41   41    0    1 -8.06801242108342881403e-01
41   41    1    1 3.25891248934106814281e+00
41   42    0    0 0.00000000000000000000e+00
41   42    0    1 -1.13620913962075950077e-01
41   42    1    1 3.14312276889235089970e-01
42   42    0    0 1.00000000000000000000e+00
42   42    0    1 -7.75726425838504885703e-01
42   42    1    1 3.13734242860591773388e+00
```

E.2 `pert.out`

E.2.1 Description

This code computes the Fourier series of the functions h, solution of the vortex equation

$$\Delta h = \frac{1}{2}\left(1 - e^{-2h}|\chi|^2\right) \tag{E.2}$$

(for more details, see chapters 4 and 5). As input this program needs the flux number, the number of orders in the bradlow parameter expansion that you want to compute, the number of Fourier modes that you want to compute, a directory to save the computed data, and the positions of the zeros of the Higgs field. A flag is used to know if the computation has to be started or continued from some order.

E.2.2 Input

A typical input file for this has the following format

```
Nstart
q, Norders, Nfourier, Taui, dir
zero1_X, zero1_Y
zero2_X, zero2_Y
zero3_X, zero3_Y
.
.
.
```

where:

Nstart: Integer. Where the computation has to be started. If Nstart is 0 or 1, the computation is started from the beginning. In any other case the computation starts in this order. If the computation does not starts from the beginning, the program needs the result of the computed data up to order Nstart, *with the rest of the input exactly the same as in the previous computation.*

q: Integer. The flux number.

Norders: Integer. The number of orders that you want to compute. The maximum is 51.

Nfourier: Integer. The number of Fourier modes that you compute in the convolutions.

Taui: Real. The aspect ratio of the torus sizes (value of the quotient l_2/l_1).

dir: Character string. the name of an existing directory where the output will be saved.

Zeros of the Higgs field: Real. The position of $q-1$ zeros of the Higgs field follow, the position of the remaining zero is computed automatically, so that the centre of mass of the zeros is in the correct position. The position of the zeros is specified *relative* to the size of the torus, and the origin is situated in the centre of the torus.

E.2.3 Output

In the specified directory, we get the following output files:

hf:O=<oo>:Nterm=<nn>:flux=<q>.dat: The Fourier series of the function h. <oo> is a two digit integer specifying the order, <nn> is also a two digit integer that specifies the number of Fourier modes, and <q> is a one digit integer specifying the flux.

`deltaf:O=<oo>:Nterm=<nn>:flux=<q>.dat`: The Fourier series of the function Δh. `<oo>` is a two digit integer specifying the order, `<nn>` is also a two digit integer that specifies the number of Fourier modes, and `<q>` is a one digit integer specifying the flux.

`Pf:O=<oo>:Nterm=<nn>:flux=<q>.dat`: The Fourier series of the function e^{-2h}. `<oo>` is a two digit integer specifying the order, `<nn>` is also a two digit integer that specifies the number of Fourier modes, and `<q>` is a one digit integer specifying the flux.

`factor:O=<oo>:Nterm=<nn>:flux=<q>.dat`: The value of $e^{-2h_{00}}$, where h_{00} is the $\vec{0}$ Fourier mode of h. `<oo>` is a two digit integer specifying the order, `<nn>` is also a two digit integer that specifies the number of Fourier modes, and `<q>` is a one digit integer specifying the flux.

`summary.dat`: A text file with a small summary of the computed data: flux number, positions of the zeros, number of computed Fourier modes, time of the computation, etc...

E.2.4 Example

Here a small example of the use of the program:

```
[14] user@computer:~/prg/pert % cat input.dat
0
2, 2, 12, 1.0, data
0.0, 0.25
[15] user@computer:~/prg/pert % cat input.dat| ./pert.out

 Stage 1: Reading input and needed data:
 ========================================
 Flux (a.k.a q, first chern class):    2
 Orders to compute:      2
 Number of terms used for the Fourier series:   12
 Aspect ratio of the Torus (l2/l1): 1.0000E+00

 Positions of the zeros (relative to the center of the Torus):
  0.0000000000000000000000000E+00  2.5000000000000000000000000E-01
  0.0000000000000000000000000E+00 -2.5000000000000000000000000E-01


 Stage 2: Computing h, Factor:
 ================================
Order:     2 done
[16] user@computer:~/prg/pert % ls data
deltaf:O=01:Nterm=12:flux=2.dat  hf:O=01:Nterm=12:flux=2.dat
deltaf:O=02:Nterm=12:flux=2.dat  hf:O=02:Nterm=12:flux=2.dat
factor:O=01:Nterm=12:flux=2.dat  Pf:O=02:Nterm=12:flux=2.dat
factor:O=02:Nterm=12:flux=2.dat  summary.dat
[19] alberto@computer:~/prg/pert % cat input2.dat
2
```

```
2, 5, 12, data
0.0, 0.25
[20] user@computer:~/prg/pert % cat input2.dat | ./pert.out

 Stage 1: Reading input and needed data:
 ========================================
 Flux (a.k.a q, first chern class):    2
 Orders to compute:       5
 Number of terms used for the Fourier series:   12
 Aspect ratio of the Torus (l2/l1): 1.0000E+00

 Positions of the zeros (relative to the center of the Torus):
  0.000000000000000000000000E+00  2.500000000000000000000000E-01
  0.000000000000000000000000E+00 -2.500000000000000000000000E-01

Reading order:     2

 Stage 2: Computing h, Factor:
 ================================
Order:     3 done
Order:     4 done
Order:     5 done
[21] user@computer:~/prg/pert % ls data
deltaf:O=01:Nterm=12:flux=2.dat  hf:O=01:Nterm=12:flux=2.dat
deltaf:O=02:Nterm=12:flux=2.dat  hf:O=02:Nterm=12:flux=2.dat
deltaf:O=03:Nterm=12:flux=2.dat  hf:O=03:Nterm=12:flux=2.dat
deltaf:O=04:Nterm=12:flux=2.dat  hf:O=04:Nterm=12:flux=2.dat
deltaf:O=05:Nterm=12:flux=2.dat  hf:O=05:Nterm=12:flux=2.dat
factor:O=01:Nterm=12:flux=2.dat  Pf:O=02:Nterm=12:flux=2.dat
factor:O=02:Nterm=12:flux=2.dat  Pf:O=03:Nterm=12:flux=2.dat
factor:O=03:Nterm=12:flux=2.dat  Pf:O=04:Nterm=12:flux=2.dat
factor:O=04:Nterm=12:flux=2.dat  Pf:O=05:Nterm=12:flux=2.dat
factor:O=05:Nterm=12:flux=2.dat  summary.dat
[22] user@computer:~/prg/pert % cat data/summary.dat
#
# SUMMARY of computed data. Tue Jan 23 18:37:49 2007
#
# Flux (a.k.a q, first chern class):    2
# Orders computed:       5
# Number of terms used for the Fourier series:   12
# Aspect ratio of the Torus (l2/l1): 1.0000E+00

# Positions of the zeros (relative to the center of the Torus):
  0.000000000000000000000000E+00  2.500000000000000000000000E-01
  0.000000000000000000000000E+00 -2.500000000000000000000000E-01

# Values of C (position in moduli space in the coordinates of CP):
```

```
#  -7.07106781186555233276180840E-01 -9.37559000231067024114859530E-17
#  -7.07106781186539912198441020E-01 -1.66030916847532790753176540E-16
#
# Times of the computation:
#
Reading order:     2(1.10E-02sec)
Order:     3 done (2.20E-02sec)
Order:     4 done (3.20E-02sec)
Order:     5 done (4.60E-02sec)
```

E.3 `metric.out`

E.3.1 Description

This code computes the metric in the space of solutions of the Bogomolny equations. For more details, see chapter 7. As input this program needs the flux number, the number of orders in the bradlow parameter expansion that you want to compute, the number of Fourier modes that you want to compute, a directory to save the computed data, and the positions of the zeros of the Higgs field. The computation of the metric has two different parts, and the first one requires the computation of exactly the same quantities than to solve the Bogomolny equations, that is what the program `pert` (E.2) does. This code can compute all the data from scratch of use previously computed data. Two flags are used to know if the computation has to be started or continued.

E.3.2 Input

A typical input file for this has the following format

```
Nstart1, Nstart2
q, Norders, Nfourier, dir
zero1_X, zero1_Y
zero2_X, zero2_Y
zero3_X, zero3_Y
.
.
.
```

where:

`Nstart1`: Integer. Where the first part of the computation has to be started. If `Nstart1` is 0 or 1, the computation is started from the beginning. In any other case the computation starts in this order. If the computation does not starts from the beginning, the program needs the result of the computed data up to order `Nstart`, *with the rest of the input exactly the same as in the previous computation.* In this case these previously computed data can be the ones computed with the program `pert`.

`Nstart2`: Integer. Where the second part of the computation has to be started. If `Nstart2` is 0 or 1, the computation is started from the beginning. In any other case the computation starts in this order. If the computation does not starts from the beginning, the program needs the result of the computed data up to order Nstart, *with the rest of the input exactly the same as in the previous computation.*

q: Integer. The flux number.

Norders: Integer. The number of orders that you want to compute. The maximum is 51.

Nfourier: Integer. The number of Fourier modes that you compute in the convolutions.

dir: Character string. the name of an existing directory where the output will be saved.

Zeros of the Higgs field: Real. The position of $q-1$ zeros of the Higgs field follow, the position of the remaining zero is computed automatically, so that the centre of mass of the zeros is in the correct position. The position of the zeros is specified *relative* to the size of the torus, and the origin is situated in the centre of the torus.

E.3.3 Output

In the specified directory, we get the following output files:

hf:O=<oo>:Nterm=<nn>:flux=<q>.dat: The Fourier series of the function h. <oo> is a two digit integer specifying the order, <nn> is also a two digit integer that specifies the number of Fourier modes, and <q> is a one digit integer specifying the flux.

deltaf:O=<oo>:Nterm=<nn>:flux=<q>.dat: The Fourier series of the function Δh. <oo> is a two digit integer specifying the order, <nn> is also a two digit integer that specifies the number of Fourier modes, and <q> is a one digit integer specifying the flux.

Pf:O=<oo>:Nterm=<nn>:flux=<q>.dat: The Fourier series of the function e^{-2h}. <oo> is a two digit integer specifying the order, <nn> is also a two digit integer that specifies the number of Fourier modes, and <q> is a one digit integer specifying the flux.

hdotf:<i>:O=<oo>:Nterm=<nn>:flux=<q>.dat: The Fourier series of the function $\mathcal{H}_i$. <i> is a one digit integer specifying which function, <oo> is a two digit integer specifying the order, <nn> is also a two digit integer that specifies the number of Fourier modes, and <q> is a one digit integer specifying the flux.

factor:O=<oo>:Nterm=<nn>:flux=<q>.dat: The value of $e^{-2h_{00}}$, where h_{00} is the $\vec{0}$ Fourier mode of h. <oo> is a two digit integer specifying the order, <nn> is also a two digit integer that specifies the number of Fourier modes, and <q> is a one digit integer specifying the flux.

Bf:<i>:O=<oo>:Nterm=<nn>:flux=<q>.dat: The value of $\mathcal{H}_i(\vec{0})$, the $\vec{0}$ Fourier mode of $\mathcal{H}_i$. <i> is a one digit integer specifying which function, <oo> is a two digit integer specifying the order, <nn> is also a two digit integer that specifies the number of Fourier modes, and <q> is a one digit integer specifying the flux.

metric:O=<oo>:Nterm=<nn>:flux=<q>.dat: Complex. The metric. The metric is as q complex numbers with format ES33.25. An example to read the stored data and save it in the two dimensional array g(:,:) is:

```
Open (Unit = 99, File = 'metric:O=34:Nterm=16:flux=2.dat')
Do I = 1, q
  Read(99,'(100ES33.25)')(g(I,J), J = 1, q)
End Do
Close (99)
```

The name conventions for the files are the same as before: `<oo>` is a two digit integer specifying the order, `<nn>` is also a two digit integer that specifies the number of Fourier modes, and `<q>` is a one digit integer specifying the flux.

`summary_metric.dat`: A text file with a small summary of the computed data: flux number, positions of the zeros, number of computed Fourier modes, time of the computation, etc...

E.3.4 Example

A small example that shows some possibilities of the program. It uses the output of the example detailed in (E.2):

```
[46] user@computer:~/prg/metric % cat point:0000.dat
5, 0
2, 7, 12,  data
0.0, 0.25
[47] user@computer:~/prg/metric % cat point:0000.dat | ./metric.out

 Stage 1: Reading input and needed data:
 =======================================
 Flux (a.k.a q, first chern class):    2
 Orders to compute:       7
 Number of terms used for the Fourier series:   12

 Positions of the zeros (relative to the center of the Torus):
  0.0000000000000000000000000E+00  2.5000000000000000000000000E-01
  0.0000000000000000000000000E+00 -2.5000000000000000000000000E-01

Reading order:     2
Reading order:     3
Reading order:     4
Reading order:     5

 Stage 2: Computing h, Factor:
 ===============================
Order:     6 done
Order:     7 done

 Stage 3: Computing \dot H_i, B_i:
 ===================================
Order:     2 done
```

```
Order:     3 done
Order:     4 done
Order:     5 done
Order:     6 done
Order:     7 done

 Stage 4: Computing the metric:
 ================================
Order:     1 done
Order:     2 done
Order:     3 done
Order:     4 done
Order:     5 done
Order:     6 done
Order:     7 done
[48] user@computer:~/prg/metric % cat point2:0000.dat
7, 7
2, 10, 12,  data
0.0, 0.25
[49] user@computer:~/prg/metric % cat point2:0000.dat | ./metric.out

 Stage 1: Reading input and needed data:
 =======================================
 Flux (a.k.a q, first chern class):    2
 Orders to compute:      10
 Number of terms used for the Fourier series:   12

 Positions of the zeros (relative to the center of the Torus):
  0.0000000000000000000000000E+00  2.5000000000000000000000000E-01
  0.0000000000000000000000000E+00 -2.5000000000000000000000000E-01

Reading order:     2
Reading order:     3
Reading order:     4
Reading order:     5
Reading order:     6
Reading order:     7

 Stage 2: Computing h, Factor:
 ===============================
Order:     8 done
Order:     9 done
Order:    10 done

 Stage 3: Computing \dot H_i, B_i:
 ===================================
Reading order:     2
```

```
Reading order:     3
Reading order:     4
Reading order:     5
Reading order:     6
Reading order:     7
Order:     7 done
Order:     8 done
Order:     9 done
Order:    10 done

 Stage 4: Computing the metric:
 ================================
Order:     7 done
Order:     8 done
Order:     9 done
Order:    10 done
[51] user@computer:~/prg/metric % cat data/summary_metric.dat
#
# SUMMARY of computed data. Mon Jan 15 12:20:55 2007
#
# Flux (a.k.a q, first chern class):    2
# Orders computed:      10
# Number of terms used for the Fourier series:   12

# Positions of the zeros (relative to the center of the Torus):
  0.0000000000000000000000000E+00  2.5000000000000000000000000E-01
  0.0000000000000000000000000E+00 -2.5000000000000000000000000E-01

# Values of C (position in moduli space in the coordinates of CP):
#  -7.0710678118655523327618084E-01 -9.3755900023106702411485953E-17
#  -7.0710678118653991219844102E-01 -1.6603091684753279075317654E-16
#
# Times of the computation (PART I):
#
Reading order:     2(2.80E-02sec)
Reading order:     3(2.20E-02sec)
Reading order:     4(1.40E-02sec)
Reading order:     5(8.00E-03sec)
Reading order:     6(7.00E-03sec)
Reading order:     7(9.00E-03sec)
Order:     8 done (2.35E-01sec)
Order:     9 done (3.43E-01sec)
Order:    10 done (5.04E-01sec)
#
# Times of the computation (PART II):
#
Reading order:     2(4.00E-03sec)
```

```
Reading order:     3(5.00E-03sec)
Reading order:     4(5.00E-03sec)
Reading order:     5(5.00E-03sec)
Reading order:     6(5.00E-03sec)
Reading order:     7(5.00E-03sec)
Order:     7 done (1.24E-01sec)
Order:     8 done (1.37E-01sec)
Order:     9 done (1.52E-01sec)
Order:    10 done (1.68E-01sec)
#
# Times of the computation (PART III):
#
Order:     7 done (5.70E-01sec)
Order:     8 done (6.57E-01sec)
Order:     9 done (7.42E-01sec)
Order:    10 done (8.28E-01sec)
```

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